NOTES

FUNDAMENTALS OF TRANSMISSION

~~Carlton E. Tucker~~



# NOTES ON

# FUNDAMENTALS OF TELEPHONE TRANSMISSION

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# FUNDAMENTALS OF TELEPHONE TRANSMISSION

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## FUNDAMENTALS OF TELEPHONE TRANSMISSION

### INTRODUCTION

The problems of the telephone transmission are usually treated on the basis of the transmission of alternating currents. In presenting the principles of alternating current transmission and their practical application to telephone problems, it is of advantage to start with the distribution of direct currents in electrical circuits. This is due to the fact that the same general relations hold for both alternating and direct currents. The presentation of these principles for direct currents is simpler because the two factors, frequency and time relation of current to voltage, are not involved. Furthermore, the generalizations of those relations to include these two factors can be easily made and the principles of direct current flow in circuits extended to alternating currents. This presentation thus deals, first, with the derivation of the useful electrical relations for direct currents, second, with the fundamental disctinction between alternating and direct currents, and third, with the generalization of these relations to cover alternating currents.

In what follows, a knowledge of the fundamentals of electricity is assumed. A presentation of the material covered by this assumption is given in a book by W. H. Timbie, entitled "Elements of Electricity" (published 1910 by John Wiley & Sons Chapters 1 to X1 inclusive, and Chapter XV. In this book and also in this set of notes, the use of higher mathematics has been avoided, the treatment being limited practically to the use of algebra.

## I. Direct Currents.

The principles governing the flow of direct currents in circuits can be expressed in three laws, one known as Ohm's Law and two others known as Kirchoff's Laws. The determination of the distribution of direct current, voltage and power in a circuit is a matter of the application of these relations.

### A. Networks.

#### 1. Ohm's Law.

For a simple series circuit containing an electromotive force of voltage E and a resistance R, Ohm's Law states that the current I which will flow through the resistance R is equal to the voltage divided by the resistance.

Thus, referring to figure 1,

$$I = \frac{E}{R} \qquad (1)$$

This may also be expressed in the form that the voltage across the source E is equal to the voltage drop V across the resistance, thus,

$$E = V = IR \qquad (2)$$

In an electrical circuit containing several resistances in series, the total resistance is the sum of these resistances. Thus, for the circuit in figure 2,

$$R = R_1 + R_2 \qquad (3)$$

The current in this circuit then is

$$I = \frac{E}{R_1 + R_2} \qquad (4)$$

and

$$E = IR_1 + IR_2 \qquad (5)$$

From this it follows that the voltage drop across $R_2$ is

$$E - IR_1 = V_2 = IR_2 \qquad (6)$$

This last equation brings out the idea that the voltage across the points a and c, by the way of b, may be expressed as the algebraic sum of the electromotive force and the potential drops, where the potential drops in the direction of the current are opposed to the electromotive force producing the current. Also in the equation $V_2 = IR_2$, Ohm's law is applied to a part of a circuit to give the relation that the potential drop across any part of a circuit is equal to the product of the current through that part of the circuit by the resistance of that part of the circuit.

For the reciprocal of resistance, the term conductance is often used ($\frac{1}{R} = G$). Thus, equation 1 could be written,

$$I = EG \qquad (7)$$

2. Kirchoff's Laws.

Kirchoff's first law states that -

At any point in a circuit there is as much current flowing to the point as there is away from it.

Thus, at point a in figure 3,

$$I_1 = I_2 + I_3 \qquad (8)$$

and at point b in figure 4,

$$I_2 = I_4 + I_5 \qquad (9)$$

Kirchoff's second law states that -

In any closed electric circuit the algebraic sum of the electromotive forces and potential drops is equal to zero. If there is no electromotive force in the circuit the sum of the potential drops in one direction is equal to the sum of those of opposite direction.

Thus, for figure 1,

$$E - IR = 0 \tag{10}$$

For circuit a d e of figure 4,

$$E - I_1 R_1 - I_3 R_3 = 0 \tag{11}$$

For circuit a b c d of figure 4,

$$I_2 R_2 + I_5 R_5 = I_3 R_3 \tag{12}$$

and for circuit b f c of figure 4,

$$I_4 R_4 = I_5 R_5 \tag{13}$$

3. Application of Kirchoff's Laws.

The application of these laws to any particular network having more than one closed mesh is a matter of algebraic solution of simultaneous equations. Referring to a two mesh network such as shown by figure 3, assume some direction for the currents through the resistances. The direction assumed is immaterial, but in the solution of complicated networks, errors are often avoided by assuming one direction throughout. It is usual to use a clockwise direction. Then, for mesh a d e,

$$E - I_1 R_1 - I_3 R_3 = 0 \tag{14}$$

and for mesh a b c d

$$I_2 R_2 = I_3 R_3 \tag{15}$$

and using Kirchoff's first law

$$I_1 = I_2 + I_3 \qquad (16)$$

There are now three simultaneous equations which can be solved for $I_1$ $I_2$ and $I_3$.

Substituting the value of $I_2 = I_1 - I_3$ obtained from equation (16) in equation (15) gives

$$(I_1 - I_3) R_2 = I_3 R_3 \qquad (17)$$

from which

$$I_3 = I_1 \frac{R_2}{R_2 + R_3} \qquad (18)$$

Substituting equation (18) in equation (14) gives

$$E - I_1 R_1 - I_1 \frac{R_2 R_3}{R_2 + R_3} = 0 \qquad (19)$$

or

$$I_1 = \frac{E}{R_1 + \frac{R_2 R_3}{R_2 + R_3}} \qquad (20)$$

Placing the value of $I_3$ as given by equation (18) in (15)

$$I_2 = I_1 \frac{R_3}{R_2 + R_3} \qquad (21)$$

Collecting these equations for the several currents,

$$I_1 = \frac{E}{R_1 + \frac{R_2 R_3}{R_2 + R_3}} \qquad (20)$$

$$I_2 = \frac{R_3}{R_2 + R_3} I_1 \qquad (21)$$

$$I_3 = \frac{R_2}{R_2 + R_3} I_1 \qquad (18)$$

The first of these three equations is an expression for Ohm's law in which the denominator of the right hand term of the equation is the total resistance which the network presents to the terminals of the electromotive force. From the circuit it is seen that this resistance is made of $R_1$ in series with the combination of $R_2$ and $R_3$ in parallel.

Comparing equation (4) with the above expression for $I_1$ it will be seen that the same current could be produced by impressing the electromotive force E on a simple series circuit having two resistances $R_1$ and $R_2{}^1$, where $R_2{}^1 = \dfrac{R_2 R_3}{R_2 + R_3}$.

Quite in general, it can be shown, that as far as the external circuit is concerned a number of resistances in parallel can be replaced by a single resistance.

$\dfrac{R_2 R_3}{R_2 + R_3}$ is the expression then for the two resistances in parallel. The general relation has thus been deduced that -

& The resistance offered by two resistances in parallel is equal to the product of the two resistances divided by their sum.

The above expression for the resistances $R_2$ and $R_3$ in parallel can be simplified by introducing the conductances $G_2 = \dfrac{1}{R_2}$ and $G_3 = \dfrac{1}{R_3}$. It is readily seen that -

$$R_2{}^1 = \frac{R_2 R_3}{R_2 + R_3} = \frac{1}{G_2 + G_3} \tag{22}$$

Calling $G = \dfrac{1}{R_2{}^1}$, the conductance of the combination, it follows from equation (22)

$$G = G_2 + G_3 \tag{23}$$

This relation can be stated generally that -

Where several resistances are in parallel the total conductance of the combination is the sum of the separate conductances.

In cases where there are a number of resistances in parallel, it is often of advantage to solve for the total conductance of the circuit. The total resistance is always the reciprocal of the total conductance.

From the equations (18) and (21) for the values of $I_3$ and $I_2$, the respective currents through $R_3$ and $R_2$, the following relation may be deduced from the division of the total current flowing to a combination of two resistances in parallel.

b. For two resistances $R_2$ and $R_3$ in parallel, the ratio of the current through $R_2$ to the total current flowing to the combination is equal to the ratio of the resistance $R_3$ to the sum of the two resistances. Note that the division of current between the branches of a parallel circuit is independent of the amount of resistance external to the branched circuit. For this reason the resistance $R_1$ does not appear in equations (18) and (21).

These two relations a and b deduced from the application of Kirchoff's laws may be used in place of the simultaneous equations to determine the current distribution in a circuit. Referring to figure 4, the current distribution for the circuit shown will be determined by both means.

For circuit a d e, $I_1 R_1 + I_3 R_3 = E$ (24)

For circuit a b c d, $I_2 R_2 + I_5 R_5 = I_3 R_3$ (25)

$$I_4 R_4 = I_5 R_5 \quad (26)$$

$$I_2 + I_3 = I_1 \quad (27)$$

$$I_4 + I_5 = I_2 \qquad (28)$$

Five independent equations are required to solve for the five currents. Other equations can be used if desired to replace some of the above.

For circuit a b c d e

$$I_1 R_1 + I_2 R_2 + I_5 R_5 = E \qquad (29)$$

For circuit a b f c d

$$I_2 R_2 + I_4 R_4 = I_3 R_3 \qquad (30)$$

Equation (29) is not independent of equations (24) and (25) since it can be obtained directly from them by substituting in equation (24) the value of $I_3 R_3$ given by equation (25). Likewise (30) is not independent of (25) and (26). Equation (29) could be used instead of either (24) or (25) and (30) instead of either (25) or (26).

Substituting the values of $I_3$ and $I_5$ given by equations (27) and (28) in equations (24), (25) and (26) gives

$$I_1 R_1 + (I_1 - I_2) R_3 = E \qquad (31)$$

$$I_2 R_2 + (I_2 - I_4) R_5 = (I_1 - I_2) R_3 \qquad (32)$$

$$I_4 R_4 = (I_2 - I_4) R_5 \qquad (33)$$

These equations can then be solved to give the following values for $I_1$ $I_2$ and $I_4$.

$$I_1 = \frac{E}{R_1 + \dfrac{R_3\left(R_2 + \dfrac{R_4 R_5}{R_4 + R_5}\right)}{R_3 + R_2 + \dfrac{R_4 R_5}{R_4 + R_5}}} \qquad (34)$$

$$I_2 = I_1 \frac{R_3}{R_3 + R_2 + \frac{R_4 R_5}{R_4 + R_5}} \qquad (35)$$

$$I_4 = I_2 \frac{R_5}{R_4 + R_5} \qquad (36)$$

The value of $I_4$ is obtained from equation (33). This value placed in equation (32) gives the above value for $I_2$ in terms of $I_1$. This value of $I_2$ substituted in eqqation (31) permits the solution for $I_1$.

<u>From the two relations deduced from the solution of the network of figure 3 these equations (34), (35) and (36) can be written down directly.</u> Equation (36) is the expression for the portion of $I_2$ which flows through $R_4$. Similarly equation (35) is the expression for the portion of $I_1$, which flows through $R_2$ in series with the combination of $R_4$ and $R_3$ in parallel, that is, through $R_2 + \frac{R_4 R_5}{R_4 + R_5}$. Equation (34) is the expression for the current flowing from E as determined by dividing this selectromotive force by the resistance across its terminals. This resistance consists of $R_1$ in series with the combination of $R_3$ and $R_2 + \frac{R_4 R_5}{R_4 + R_5}$ in parallel.

From the equations (34), (35) and (36) the values of currents $I_3$ and $I_5$ can be obtained with the aid of equations (27) and (28). These methods can be used to solve any such network even though containing many more meshes. The first method will be referred to as the simultaneous equation method and the second as the direct method.

## 4. Networks Containing More than One E.M.F.

A network containing more than one E.M.F. such as that shown by figure 5 can be solved from equations such as those set up for network 4, the only change required in this case (where the networks are identical except for the inclusion of $E_2$ in figure 5) would be that equation (25) would be changed to include $E_2$ thus.

$$I_2 R_2 + I_5 R_5 + E_2 = I_3 R_3 \qquad (37)$$

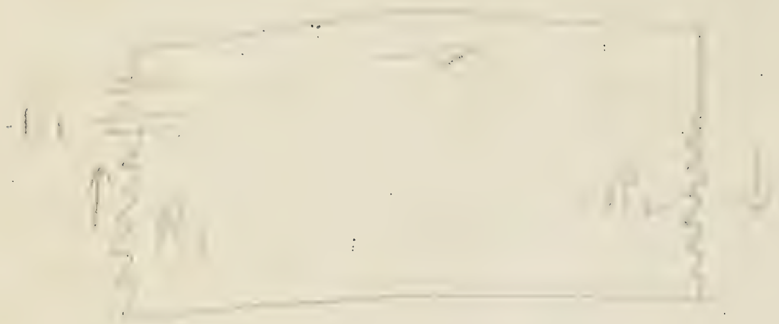
The network can also be solved for each voltage separately assuming the other voltage not present. The two currents obtained in each branch by the two separate solutions can then be combined adding them where they are in the same direction and subtracting when opposite, the resultant in the last case being in the direction of the larger component. These separate solutions may be made either by the simultaneous equation method or by the direct method.

This method of solution by superposing the results of separate determination for each E.M.F. in the circuit will apply for any number of E.M.F.'s in the circuit and for any network.

## 5. Efficiency and Maximum Power Transfer.

In an electrical circuit the usual purpose is to transmit power to a device such as a relay, motor, incandescent light or heater in order to make it perform some operation.

In the circuits which have been considered, the sources of E.M.F. have been assumed to have no internal resistances. In general, however, any source of voltage has an internal resistance A storage battery and a generator have internal resistance. The combination of the generator and the circuit, used to transmit



power from the generator to the device to be operated by the power, has resistance.

Referring to figure 6, $E_1$ and $R_1$ are used to designate respectively the voltage generated in a direct current generator and the internal resistance of the generator. $R_2$ is the resistance of the heater element of an electric flat-iron. The current delivered to the flat-iron is,

$$I = \frac{E_1}{R_1 + R_2} \qquad (38)$$

and the power $P_2$ delivered to the iron is,

$$P_2 = I^2 R_2 \qquad (39)$$

The power dissipated in the generator is,

$$P_1 = I^2 R_1 \qquad (40)$$

The efficiency F of this arrangement then, expressed as the ratio of the energy delivered to the device to be operated to the total energy generated, is,

$$F = \frac{P_2}{P_1 + P_2} = \frac{I^2 R_2}{I^2 R_1 + I^2 R_2} = \frac{R_2}{R_1 + R_2} = \frac{1}{\frac{R_1}{R_2} + 1} \qquad (41)$$

The efficiency so expressed increases as the ratio of $\frac{R_1}{R_2}$ is de- decreased; that is, as $R_2$ becomes large compared to $R_1$.

Consider now, however, the amount of power delivered to $R_2$. The expression for this as derived from equations (38) and (39) is,

$$P_2 = \frac{E_1^2 R_2}{(R_1 + R_2)^2} \qquad (42)$$

The total power generated is,

$$P_1 + P_2 = \frac{E_1^2 (R_1 + R_2)}{(R_1 + R_2)^2} = \frac{E_1^2}{R_1 + R_2} \qquad (43)$$

The effect of the relative magnitudes of $R_2$ and $R_1$ on the values of F, $P_2$ and $P_1 + P_2$ as given in equations (41), (42) and (43) is shown in the following table:

| Value of $R_2$ | $P_2 = \frac{E_1^2 R_2}{(R_1 + R_2)^2}$ | $P_1 + P_2 = \frac{E_1^2}{R_1 + R_2}$ | $F = \frac{100}{\frac{R_1}{R_2} + 1}$ % |
|---|---|---|---|
| $2.00 R_1$ | $.2222 \frac{E_1^2}{R_1}$ | $.3333 \frac{E_1^2}{R_1}$ | 66.67% |
| $1.10 R_1$ | $.2494 \frac{E_1^2}{R_1}$ | $.4762 \frac{E_1^2}{R_1}$ | 52.38% |
| $1.02 R_1$ | $.24998 \frac{E_1^2}{R_1}$ | $.4950 \frac{E_1^2}{R_1}$ | 50.50% |
| $1.00 R_1$ | $.2500 \frac{E_1^2}{R_1}$ | $.5000 \frac{E_1^2}{R_1}$ | 50.00% |
| $.98 R_1$ | $.24997 \frac{E_1^2}{R_1}$ | $.5050 \frac{E_1^2}{R_1}$ | 49.50% |
| $.90 R_1$ | $.2493 \frac{E_1^2}{R_1}$ | $.5263 \frac{E_1^2}{R_1}$ | 47.37% |
| $.50 R_1$ | $.2222 \frac{E_1^2}{R_1}$ | $.6667 \frac{E_1^2}{R_1}$ | 33.33% |

It will be noted that the expression $\frac{E_1^2}{R_1}$, which occurs in all the terms in the second and third columns of the above table, is the value of the total power generated and dissipated in the

generator when its terminals are short circuited. As the value of $R_2$ is made smaller, the values in column three approach the power generated on short circuit as a limit.

These values are plotted as curves on the attached Drawing No. 910-27.

From this table and the curves it is seen that while the efficiency of the system as expressed by equation (41) increases as $R_2$ becomes greater the power delivered to $R_2$ is a maximum when $R_2$ is equal to $R_1$. For this condition 50 per cent. of the total power generated is delivered to the circuit connected to the generator terminals and 50 per cent. is dissipated within the generator.

In the foregoing it was assumed that the generated voltage $E_1$ remained constant as the load varied. This corresponds with actual practice as most generating devices have a definite generated voltage. It is, however, necessary to distinguish sharply between the generated voltage and the voltage which the generator presents at its terminals - the terminal voltage of the generator. The latter voltage differs from the generated voltage by the drop in the internal resistance. Referring to Figure 8 again, it is the terminal voltage:

$$E_t = E_1 - I_1 R_1 \qquad (44)$$

As this equation shows, the terminal voltage depends on the current $I_1$ and hence varies with the load.

The above equation indicates also a method for measuring the generated voltage. It shows, namely, that the generated

voltage is the same as the terminal voltage of the generator on open circuit.

The condition for maximum transfer of power may be stated as follows:

With a generating device which has a given generated voltage and has a fixed internal resistance, the maximum power is delivered from these terminals when the external resistance equals the internal resistance.

6. Efficiency of Networks.

The determination of the distribution of current in a network is of practical interest in determining the amount of current and power delivered over a network to the receiving element. In this connection it is often necessary to compare different arrangements of supplying energy to a receiving device. Assuming that the receiving device has a resistance R and that the current supplied to it over one circuit is $I_1$ and over another circuit $I_2$, the ratio of the power supplied in the two cases is:-

$$\frac{P_2}{P_1} = \frac{I_2^2}{I_1^2}\frac{R}{R} = \left(\frac{I_2}{I_1}\right)^2 \qquad (45)$$

The above equation shows that when the resistance of the device is kept fixed the ratio of the currents can be used to compare the two arrangements, since the current ratio is the square root of the power ratio.

If, however, the operating device is modified so as to make its resistance change from $R_1$ to $R_2$ then the ratio of the power supplied in the two cases would be expressed by

$$\frac{P_2}{P_1} = \frac{I_2^2}{I_1^2}\frac{R_2}{R_1} \qquad (46)$$

If it is desired to use current ratio to express the effect of the change, the equivalent current ratio for equation (46) is given by the square root of the ratio of the powers.

The current ratio or the equivalent current ratio will be used here in comparing the transmission efficiency of two circuits. The reason for doing this will become apparent when the subject of telephone lines is discussed.

7. Transition Loss

It has been shown that when a generating element having an internal resistance $R_1$ is to deliver power to a load having a resistance $R_2$, the maximum power is delivered when $R_2$ is equal to $R_1$. The reduction in the power delivered to the receiving element caused by $R_2$ being not equal to $R_1$ is referred to as the transition loss at the junction of the generating and receiving elements. The magnitude of this reduction can be indicated by the ratio of the power delivered to the receiving element when $R_2$ is not equal to $R_1$, to the power delivered when $R_2$ equals $R_1$

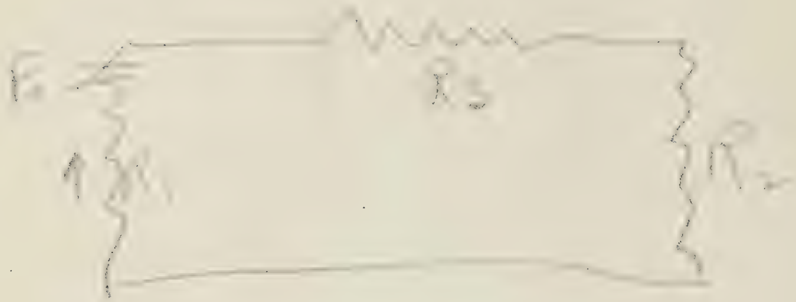
$$P_2 = \frac{E^2 R_2}{(R_1 + R_2)^2} \tag{47}$$

and the power $P_1$ delivered to $R_2$ if it were equal to $R_1$ is

$$P_1 = \frac{E^2}{4R_1} \tag{48}$$

Then

$$\frac{P_2}{P_1} = \frac{4R_1R_2}{(R_1+R_2)^2} \tag{49}$$


E
R3
R1
R2

and the measure of the effect of $R_2$ being unequal to $R_1$ on the delivered power is, on the basis of equivalent current ratio,

$$\sqrt{\frac{P_2}{P_1}} = \frac{2\sqrt{R_1 R_2}}{R_1 + R_2} = \frac{\sqrt{R_1 R_2}}{\frac{R_1 + R_2}{2}} \qquad (50)$$

Equation (50) is seen to be the ratio of the geometric to the arithmetic mean of $R_1$ and $R_2$.

8. Effect of Series Resistance.

Referring to figure 7, the effect on the current delivered to $R_2$ of inserting $R_3$ between the generator $R_1$ and the operating device $R_2$ will be determined. Without $R_3$ in the circuit the current $I_1$ through $R_2$ is

$$I_1 = \frac{E}{R_1 + R_2} \qquad (51)$$

with $R_3$ in the circuit the current $I_2$ through $R_2$ is

$$I_2 = \frac{E}{R_1 + R_2 + R_3} \qquad (52)$$

The ratio of these two currents is

$$\frac{I_2}{I_1} = \frac{\frac{E}{R_1 + R_2 + R_3}}{\frac{E}{R_1 + R_2}} = \frac{R_1 + R_2}{R_1 + R_2 + R_3} \qquad (53)$$

which may also be expressed

$$\frac{I_2}{I_1} = \frac{1}{1 + \frac{R_3}{R_1 + R_2}} \qquad (54)$$

If $R_2 = R_1$, the condition for maximum energy transfer, the equations reduce to

$$\frac{I_2}{I_1} = \frac{2R_1}{2R_1 + R_3} \tag{55}$$

$$\frac{I_2}{I_1} = \frac{1}{1 + \frac{R_3}{2R_1}} \tag{56}$$

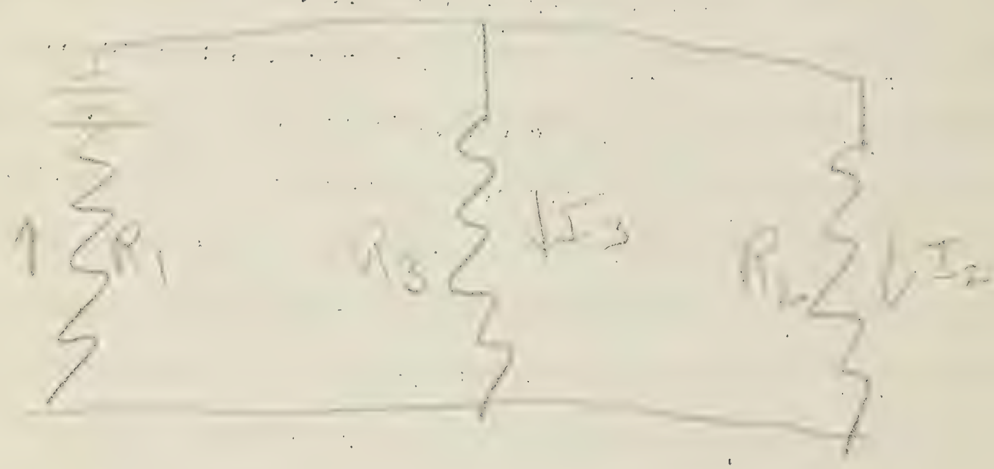
From these equations the effect on the current received over a circuit of inserting a resistance in series between given generating and receiving elements is seen. It will be noted that the received current increases as the magnitude of the resistance decreases. On the other hand if a given series resistance is to be inserted between transmitting and receiving elements which can be varied, the current ratio increases as the magnitudes of the resistances of the transmitting and receiving elements are increased. If $R_3$ is small in comparison with the total resistance of the circuit, a very good approximation to the ratio $\frac{I_2}{I_1}$ is obtained by applying the approximation:

$$\frac{1}{1 + a} = 1 - a$$

where a is small in comparison with unity. This gives

$$\frac{I_2}{I_1} = 1 - \frac{R_3}{R_2 + R_1} \tag{57}$$

that is, the current ratio is equal to unity minus the ratio of the inserted resistance to the total resistance in the circuit.



R1
r3
I3
R2
I2

Thus if this ratio is .1 the current ratio is approximately .9, exactly .909. If the ratio is .05 the approximate current ratio is .95 - the exact value is .952. From these values some idea as to the useful range of this approximation may be gained.

9. Effect of Shunt Resistance.

In Figure 8 a resistance $R_3$ is inserted in shunt between $R_1$ and $R_2$. The current $I_1$ sent through $R_2$ before $R_3$ is inserted is

$$I_1 = \frac{E}{R_1 + R_2} \tag{58}$$

After the resistance $R_3$ is placed in the circuit the current $I_2$ through $R_2$ is (see equations 20 and 21)

$$I_2 = \frac{E}{R_1 + \dfrac{R_2 R_3}{R_2 + R_3}} \times \frac{R_3}{R_2 + R_3} \tag{59a}$$

$$= \frac{ER_3}{R_1 \ (R_2 + R_3) + R_2R_3} \tag{59b}$$

$$\frac{I_2}{I_1} = \frac{\dfrac{ER_3}{R_1 \ (R_2 + R_3) + R_2R_3}}{\dfrac{E}{R_1 + R_2}} \tag{60a}$$

$$= \frac{R_3 \ (R_1 + R_2)}{R_3 \ (R_1 + R_2) + R_1R_2} \tag{60b}$$

$$= \frac{1}{1 + \dfrac{1}{R_3} \times \dfrac{R_1 R_2}{R_1 + R_2}} \tag{60c}$$

If $R_2 = R_1$ this equation becomes

$$\frac{I_2}{I_1} = \frac{1}{1 + \frac{1}{R_3}\frac{R_1}{2}} \qquad (61)$$

From these equations it is seen that a shunt resistance, to cause a small effect, whould be large compared to $R_1$ and $R_2$.

In equation (59a) the first factor, $\dfrac{E}{R_1 + \dfrac{R_2R_3}{R_2 + R_3}}$ represents the total current after the insertion of the shunt; the current is increased by the addition of the shunt. The amount of this increase depends on the magnitude of $R_1$ with respect to $R_2$ and also the magnitude of the shunt $R_3$ with respect to $R_2$. The second factor, $\dfrac{R_3}{R_2 + R_3}$, represents that portion of this total current which flows through $R_2$. This factor becomes greater, as the ratio of $R_3$ to $R_2$ is increased.

## 10. Effect of Combination of Shunt and Series Resistances.

Figure 9 shows the insertion of a combination of shunt resistance $R_3$ and series resistances $R_4$ and $R_5$. The current $I_1$ through $R_2$ before the insertion of this combination is as before

$$I_1 = \frac{E}{R_1 + R_2} \qquad (62)$$

After the combination is inserted the current $I_2$ through $R_2$ is

$$I_2 = \frac{E}{R_1 + R_4 + \dfrac{R_3(R_2 + R_5)}{R_2 + R_3 + R_5}} \times \frac{R_3}{R_2 + R_3 + R_5} \qquad (63a)$$

$$= \frac{ER_3}{(R_1+R_4)(\underset{+R_3}{R_2+R_5}) + R_3(R_2+R_5)} \tag{63b}$$

$$\frac{I_2}{I_1} = \frac{R_3\ (R_1+R_2)}{(R_1+ R_4)(R_2+ R_3+R_5) + R_3(R_2+ R_5)} \tag{63c}$$

If $R_4 = R_5$

$$\frac{I_2}{I_1} = \frac{R_3(R_1+R_2)}{(R_1+ R_4)(R_2+R_3+R_4) + R_3(R_2+R_4)} \tag{64}$$

If $R_1 = R_2$ and $R_4 = R_5$

$$\frac{I_2}{I_1} = \frac{2R_1R_3}{(R_1+R_4)(R_1+R_4+2R_3)} \tag{65}$$

## 11. Pollard's Theorem

Equations similar to the above can be derived to determine the effect on the received current of introducing series and shunt elements into networks of greater complication than those which have been just considered. Referring to figure 4, the effect of introducing $R_2$ into the network on the current $I_4$ sent through $R_4$ can be determined by computing the current $I_4$ for the conditions of $R_2$ in and not in the circuit. This would not be a very involved process for this circuit but circuits are often encountered containing many more meshes than the circuit of figure 4. For such cases, when it is desired to study the effect on the received current of a change at one point in the circuit, a relation designated as Pollard's theorem is very useful in reducing the amount of work required.

Referring to Figure 10, the circuit shown will be considered to determine the effect of inserting the resistance $R_2$ on the current through $R_5$. Pollard's theorem as applied to this circuit states that -

In computing the current $I_2$ the portion of the circuit to the left of the terminals, a b, can be replaced by a voltage V in series with a resistance R'. The voltage V is the voltage produced across the terminals a b when these terminals are open. The resistance R' is equal to the resistance of that portion of the circuit to the left of terminals a b as measured across these terminals.

This replacement is shown in Figure 11. The voltage produced across terminals a b when these terminals are open is

$$V = \frac{ER_3}{R_1 + R_3} \tag{66}$$

The resistance across these terminals is

$$R' = \frac{R_1 R_3}{R_1 + R_3} \tag{67}$$

Considering Figure 10 the current through $R_5$ is

$$I_2 = \frac{E}{R_1 + \frac{R_3 (R_2 + R_5)}{R_2 + R_3 + R_5}} \times \frac{R_3}{R_2 + R_3 + R_5} \tag{68a}$$

$$= \frac{ER_3}{R_1 (R_2 + R_3 + R_5) + R_3 (R_2 + R_5)} \tag{68b}$$

Considering Figure 11 the current through $R_5$

$$I_2 = \frac{V}{R' + R_2 + R_5} \tag{69a}$$

$$= \frac{\dfrac{ER_3}{R_1 + R_3}}{\dfrac{R_1R_3}{R_1+R_3} + R_2 + R_5} \qquad (69b)$$

$$= \frac{ER_3}{R_1R_3+(R_1+R_3)(R_2+R_5)} \qquad (69c)$$

$$= \frac{ER_3}{R_1(R_2+R_3+R_5)+R_3(R_2+R_5)} \qquad (69d)$$

Equation (69d) gives the same current through $R_5$ in Figure 11 as equation (68b) for Figure 10.

If it is desired to determine the effect on the current $I_2$ of inserting $R_2$ in the circuit of Figure 10, this may be done by using equations (66) and (67) and the circuit of Figure 11.

Thus if $I_2'$ is the current without $R_2$ and $I_2''$ the current with $R_2$ in the circuit:

$$I_2' = \frac{V}{R' + R_5} \qquad (70)$$

$$I_2'' = \frac{V}{R' + R_2 + R_5} \qquad (71)$$

$$\frac{I_2''}{I_2'} = \frac{R' + R_5}{R' + R_2 + R_5} \qquad (72)$$

Since V has dropped out of equation (72) it is not necessary to evaluate it in order to determine the effect of $R_2$ on $I_2$. It is necessary however to evaluate R'.

Consider now the application of Pollard's theorem to the circuit shown in figure 12, when it is desired to know the effect of the insertion of $R_2$ on $I_4$, the current through $R_4$. As above, the portion of the circuit to the left of terminals a b may be replaced by V and R' as shown in figure 13. Referring to the portion of the circuit to the right of terminals c d the ratio of the current $I_4$ to $I_2$ is determined by the magnitudes of $R_4$ and $R_5$.

$$I_4 = I_2 \frac{R_5}{R_4 + R_5} \tag{36}$$

Thus any change in $I_2$ will produce a directly proportional change in $I_4$. Therefore the change in $I_4$ may be determined by the change in $I_2$. The portion of the circuit to the right of terminals c d in figure 12 can be replaced by R", where $R'' = \frac{R_4 R_5}{R_4 + R_5}$ as shown in figure 13; the portion to the left by R' given in equation (67).

The effect of $R_2$ on the current in $R_4$ is then

$$\frac{I_4''}{I_4'} = \frac{I_2''}{I_2'} = \frac{R' + R''}{R' + R'' + R_2} \tag{73}$$

It will be noted that R' and R" are the resistances of the networks on each side of $R_2$ measured from the terminals to which $R_2$ is connected.

By these means the effect of a change at one point X in a complex circuit on the current at some other point Y of the circuit can be investigated, by replacing the portions of the circuit to each side of the point X by the respective resistances of these portions as was done in figure 13. This procedure will apply for the determination of the effect of a series element, a shunt elementor any combination of them.

## 12. Transfer Resistance.

Referring to figure 3 the value of the current through the receiving element $R_2$ is given by equation (59b) as

$$I_2 = \frac{ER_3}{R_1 (R_2+R_3) + R_2R_3} \qquad (59b)$$

$$= \frac{E}{R_1 + R_2 + \frac{R_1R_2}{R_3}} \qquad (59c)$$

The denominator of this equation is referred to as the "transfer resistance" of the network of figure 3. This transfer resistance divided into the voltage across the sending terminals e g of the network gives the current through the receiving element $R_2$ of the network.

The transfer resistance of any network with respect to some element in the network and a pair of terminals to which a source of voltage may be connected is defined as the factor by which this voltage must be divided to give the current through the element.

## 13. Equivalent Networks.

The definition of transfer resistance has brought up the idea that a network can be treated as an arrangement of resistances having four available terminals, the two to which the source of voltage is connected and the two to which the receiving element is connected. In many cases of complicated networks the interest is in the power delivered through the network, or with a fixed network connecting a generating and a receiving element it is desired to determine the effect of changing one of these elements. In such cases

it is possible and often convenient to replace the complex network by a simpler network which, in so far as the external circuit to which it is connected is concerned, is identical in its action to the more complex structure.

Figure 14 represents a complex combination of connected resistances, the available terminals of which are 1, 2, 3 and 4. Such a combination can be replaced by a simpler network provided that when the same generating and receiving elements are connected to either network, the following three conditions hold:

a. The current taken from the generating element is the same.

b. The voltage across the receiving terminals on open circuit is the same.

c. The resistance of the network measured across the receiving terminals is the same.

These three conditions can be met by a simple network consisting of three resistances, provided these three resistances are not all in series or all in parallel. Such an arrangement of three resistances is shown in figure 15. The magnitudes of these three resistances can be determined from the impositions of conditions a, b and c or their equivalents.

Three other requirements that are often used to determine the equivalence of two networks are -

1. The resistance across terminals 12 with 34 open,

2. across 12 with 34 short circuited and

3. across 34 with 12 open circuited. It can be shown that if a network meets these three requirements it also meets the three a,b and c, given above. As a matter of fact any three requirements involving all the four available terminals can be used to determine the equivalence of two networks.

To determine the values of a, b and c of the network of figure 15 to make it equivalent to the network of figure 14, it is necessary only to solve the three equations corresponding to the three requirements set up at the beginning of the last paragraph. These equations are as follows:

(1) The resistance $R_1$ across terminals 12 with 34 open

$$R_1 = a + c \tag{74}$$

(2) The resistance $R_2$ across terminals 12 with 34 short circuited

$$R_2 = a + \frac{b\ \ c}{b + c} \tag{75}$$

(3) The resistance $R_3$ across terminals 34 with 12 open

$$R_3 = b + c \tag{76}$$

Assuming the values of $R_1$, $R_2$ and $R_3$ to be determined either by measurement or computation from the network of figure 14, the values of a, b and c in terms of $R_1$, $R_2$ and $R_3$ are as follows:

$$c = \sqrt{(R_1 - R_2)R_3} \tag{77}$$

$$a = R_1 - c \tag{78}$$

$$b = R_3 - c \tag{79}$$

## 14. Wheatstone Bridge:

The Wheatstone bridge circuit shown in figure 16 is a very useful and interesting network and is one which has many applications in telephony aside from that as a means of measuring resistance. In its use as a bridge it is known that if $R_1'$ is the galvanometer, a voltage E in series with $R_1$ will send no current through the galvano-

meter if a proper adjustment is made of the values of the resistces $R_2$, $R_3$, $R_2'$ and $R_3'$.

If the current $I_1'$ through $R_1'$ is zero then by Kirchoff's first law applied to points b and d

$$I_2 = I_3 \tag{80}$$

and

$$I_2' = I_3' \tag{81}$$

By Kirchoff's second law applied to circuit a b d

$$I_2' \; R_2' = I_3 R_3 \tag{82}$$

or

$$\frac{I_2'}{I_3} = \frac{R_3}{R_2'} \tag{83}$$

and applied to circuit b c d

$$I_3' \; R_3' = I_2 \; R_2 \tag{84}$$

or

$$\frac{I_3'}{I_2} = \frac{R_2}{R_3'} \tag{85}$$

Then applying equations 80 and 81 to equations 83 and 85

$$\frac{R_3}{R_2'} = \frac{R_2}{R_3'} \tag{86-a}$$

or

$$R_3' \; R_3 = R_2 \; R_2' \tag{86-b}$$

For no current through $R_1'$ and a voltage in series with $R_1$ the condition to be met by the other four resistances is expressed by equation 86-a. It can be similarly shown that for a

voltage in series with $R_1'$ no current is sent through $R_1$ if the other four resistances meet the requirement of equation 86-a. It is seen then that the transfer resistance in both directions between $R_1$ and $R_1'$ is infinite.

Figure 17 shows the circuit of figure 16 drawn in a slightly different way, the resistances, however, occupy the same positions in the two circuits relative to each other. In the circuits of figure 17 point d of the bridge occupies a symmetrical position with respect to points a, b and c; thus $R_1$, $R_2$ and $R_3$ are symmetrically located with respect to d and so are $R_1'$ $R_2'$ and $R_3'$.

Similarly, the relative position of $R_2$ to $R_2'$ and of $R_3$ to $R_3'$ is the same as that of $R_1$ to $R_1'$ . From this symmetrical arrangement it is seen that the condition for infinite transfer resistance between $R_2$ and $R_2'$ is that the other four resistances bear to each other a relation corresponding to equation 86-a which is

$$\frac{R_3}{R_1'} = \frac{R_1}{R_3'} \qquad (87\text{-a})$$

$$R_3' \; R_3 = R_1 \; R_1' \qquad (87\text{-b})$$

Likewise the condition for infinite transfer resistance between $R_3$ and $R_3'$ is that

$$\frac{R_1}{R_2'} = \frac{R_2}{R_1'} \qquad (88\text{-a})$$

$$R_1' \; R_1 = R_2 \; R_2' \qquad (88\text{-b})$$

If six resistances are arranged in the circuit of fig.17 and are proportioned to have values to fulfill the following equation

$$R_1 R_1' = R_2 R_2' = R_3 R_3', \qquad (89)$$

a voltage placed in series with one resistance will not send current through the other resistance grouped with it in equation 89. It will be noted that the two resistances multiplied together in the equation 89 are the two which do not have a common terminal in the circuit of figure 17.

Furthermore, it can be shown that if a voltage is placed in series with $R_1$ and if,

$$R_1 R_1' = R_2 R_2', \qquad (90).$$

the current through $R_3$ is independent of the magnitude of $R_3'$ and vice versa. Similarly a voltage in series with any one of the four resistances in equation 90 will produce a current in $R_3$ the magnitude of which is independent of $R_3'$.

If two of the three products $R_1 R_1'$, $R_2 R_2'$, $R_3 R_3'$ are equal to each other, a voltage placed in series with any one of the four resistances in these two products will send current through one resistance of the third product, the magnitude of which is independent of the other resistance of that third product. These relations can be deduced by applying Kirchoff's laws to the circuit of figure 17 and solving the resulting equation.

## B. MULTI-SECTION UNIFORM NETWORKS

Consider now a network made up of a very large number of equal networks connected together as shown in figure 18, at one end of which is connected a source of electromotive force E, which has an internal resistance C.

### 1. Characteristic Resistance:

In studying such a network, it is necessary to know the resistance $R_0$ which it presents across the terminals x-y. If there are an infinite number of uniform sections to thr right of x-y, then the line may be cut at m-n or p-q, and the resistance will also be $R_0$ because the effect of reducing an infinite number by one or two is negligible. If the resistance across terminals x-y equals that across m-n and each equals $R_0$, the value of $R_0$ can be determined by using the network shown in figure 19. In this circuit

$$R_0 = \frac{a}{2} + \frac{(R_0 + a/2)\, b}{R_0 + a/2 + b} \tag{91}$$

From this $R_0 = \sqrt{ab + \frac{a^2}{4}}$ (92)

<u>$R_0$ is designated the "characteristic resistance" of this network and may be defined as the resistance presented by a line consisting of an infinite number of uniform networks</u>

For maximum power from E to the line, $R_0$ should be equal to the internal resistance C, of E. If this is the case, then the resistance at m n or p q, looking toward the left will also be $R_0$. Therefore, if the line of networks is cut at the

junction of any two sections, the resistance in the two directions will be each equal to $R_0$. Furthermore, if it is desired to study the action of one of the uniform sections without considering a large number of sections, the line shown in figure 18 may be cut at m-n and the line of sections to the right replaced by $R_0$. Section (1) is then placed between C, which has been made equal to $R_0$, and terminals m-n, the resistance between which is also equal to $R_0$. The resistances across the terminals of the section (1) in both directions are then the same as it if formed one of an infinite number of sections all equal to section (1). It may, therefore, be treated as if it formed one section of a line of an infinite number in series.

2. Current Distribution:

The effect of section (1) on the current $I_2$, flowing through the terminals m-n, with respect to the current, $I_1$, flowing through terminals x-y, may be determined from figure 20.

$$\frac{I_2}{I_1} = \frac{b}{a/2 + b + R_0} \qquad (93)$$

This ratio gives the effect of any one section in diminishing the current set up by the voltage E in the line of networks. The effect of two networks would be the square of this ratio, the effect of three sections the third power of the ratio, and so on. The effect of inserting any number of such sections between two resistances $R_0$ is likewise given by the ratio raised to the power determined by the number of sections inserted.

3. Attenuation Constant:

Rather than obtain powers of the current ratio per section to determine the ratio of the currents at two points separated by several sections, use is made of the following:

$$I_2 = \frac{I_1}{e^{n\alpha}} \tag{94}$$

where e = 2.7183, is the base of the natural logarithims, and $\alpha$, the exponent, is called the "attenuation constant" per section. The factor n is the number of sections involved. Evaluating $\alpha$ from the ratio of $I_2$ to $I_1$ for one section, the value of $I_2$ for any number of sections n is given by equation 94.

Equation 94 may be written -

$$\frac{I_2}{I_1} = \frac{1}{e^{n\alpha}} = e^{-n\alpha} \tag{95}$$

where the minus sign indicates the reciprocal. For one section (n = 1), as shown in figure 20,

$$e^{-\alpha} = \frac{b}{a/2+b+R_o} \tag{96}$$

and

$$\alpha = \log_e \frac{a/2+b+R_o}{b} \tag{97}$$

## C. UNIFORM LINES

A uniform line for direct currents is a circuit in which the resistance of the conductors and the leakage resistance between the conductors are constant per unit length of line. Such a line can be considered as consisting of a multi-section uniform network in which each section of the network corresponds to a very short length of the line. This representation of a uniform line by a multi-section uniform network is shown by figure 21. The uniform line shown has a total series resistance per loop mile of $R_1$ and a total leakage conductance of $G_2 = \frac{1}{R_2}$ per mile. Each section of the uniform network corresponds to $\frac{1}{m}$ th of a mile, where m is a large number. The series resistance of each section of network is therefore $\frac{R_1}{m}$, and the leakage conductance $\frac{G_2}{m}$ for which the corresponding leakage resistance is $m R_2$. One of these small sections is shown in figure 22.

### 1. Characteristic Resistance:

Corresponding to the characteristic resistance of the uniform multi-section network, there is a characteristic resistance for a uniform line. The characteristic resistance of the network of figure 22 is from equation 92.

$$R_0' = \sqrt{\frac{R_1}{m} \times m R_2 + \frac{R_1^2}{4m^2}} \qquad (98\text{-a})$$

$$= \sqrt{R_1 R_2 + \frac{R_1^2}{4m^2}} \qquad (98\text{-b})$$

which may be written

$$R_0' = \sqrt{R_1 R_2 \left(1 + \frac{R_1}{4m^2 R_2}\right)} \qquad (98\text{-c})$$

With m large, as assumed above, $1 + \frac{R_1}{4m^2 R_2}$ becomes equal practically to 1 and the characteristic resistance of a uniform line is

$$R_o = \sqrt{R_1 R_2} \qquad (99\text{-a})$$

or

$$R_o = \sqrt{\frac{R_1}{G_2}} \qquad (99\text{-b})$$

It is seem from the above that the uniform multi-section network does not represent the uniform line unless the sections are very small. Consequently the equivalent network for an appreciable length of uniform line can not be obtained by simply lumping the series and shunt constants. An expression of the equivalent network of the line will be given later.

2. Attenuation Constant:

The attenuation constant for the network of figure 22, as obtained from equation 97 is

$$\alpha = \log_e \left( \frac{\frac{R_1}{2m} + m R_2 + \sqrt{R_1 R_2 \left( 1 + \frac{R_1}{4m^2 R_2} \right)}}{m R_2} \right) \qquad (100\text{-a})$$

where $R_o$ of equation 97 is expressed in terms of its value given in equation 98-c.

This may also be expressed

$$\alpha = \log_e \left( \frac{R_1}{2m^2 R_2} + 1 + \frac{1}{m R_2} \sqrt{R_1 R_2 \left( 1 + \frac{R_1}{4m^2 R_2} \right)} \right) \qquad (100\text{-b})$$

with m large, assumed, $1 + \frac{R_1}{2m^2 R_2}$ and $1 + \frac{R_1}{4m^2 R_2}$ become practically equal to 1. The expression for $\alpha$ then is

$$\alpha = \log_e\left( 1 + \frac{1}{m} \sqrt{\frac{R_1}{R_2}} \right) \qquad (101)$$

In textbooks of Algebra* the logarithmic series will be found

$$\log_e (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots.$$

Applying this series to equation 101, since $x = \frac{1}{m}\sqrt{\frac{R_1}{R_2}}$ is small if m is very large as was assumed, all terms of the series are negligible compared to the first, and the result is obtained

$$\alpha = \frac{1}{m}\sqrt{\frac{R_1}{R_2}} \qquad (102)$$

This is the attenuation in one small section representing a very short length of the uniform line. It was assumed that there were m such sections per loop mile. The attenuation, as the line is uniform, will be the same for each section. The attenuation per loop mile consequently will be m times the attenuation per section or the attenuation per mile is

$$\alpha = \sqrt{\frac{R_1}{R_2}} \qquad (103\text{-a})$$

$$\alpha = \sqrt{R_1 G_2} \qquad (103\text{-b})$$

The derivations of both the characteristic resistance and the attenuation constant have here been given by the method of approximation in order to avoid the use of higher mathematics. The expressions obtained, however, for both the characteristic resista

*NOTE: For reference see p.558 equation 3 in a book entitled, "College Algebra" by H.B.Fine (Publ. 1905 by Ginn & Co.)

and the attenuation constant of the uniform line are exact.

It is seen that the characteristic resistance is independent of the unit length taken because the product of $\frac{R_1}{m} \times mR_2$ is always $R_1 R_2$. The attenuation constant obtained from equation 103 is for the same unit length for which $R_1$ and $R_2$ are taken. So the attenuation constant for a line "$\ell$" units in length is "$\ell$" times the value given by equation 103.

$$\alpha \ell = \ell \sqrt{\frac{R_1}{R_2}} \tag{104}$$

From the expression for $R_0$ and it is evident that if the values of these quantities are known, the values of $R_1$ and $R_2$ can be derived.

$$\alpha R_0 = \sqrt{R_1 R_2 \frac{R_1}{R_2}} = R_1 \tag{105}$$

$$\frac{R_0}{\alpha} = \sqrt{\frac{R_1 R_2}{\frac{R_1}{R_2}}} = R_2 \tag{106}$$

3. Equivalent Networks:

From the above discussion of the uniform line it is evident that the equivalent network of a length of uniform line cannot be obtained by lumping the series and shunt resistances of the length of line except in the limiting case when the portion of line is very small. Any section of uniform line, (refer to figure 23) which presents four terminals can be treated in the same way as the network of fig. 14 and an equivalent network of three elements determined by solving the three equations obtained from the expression of three relations regarding the four terminals of the line. This solution for the uniform line is more involved than for a network and will

not be carried through here* although the formulae will be given for determining the three elements of figure 24 which is an equivalent network of the line of figure 23.

The concise and exact expression of these relations involves the use of "hyperbolic functions" which are analogous to "trigonometric functions" and are given in similar tables. Hyperbolic functions** are distinguished from the corresponding trigonometric functions by adding the letter "h" to the trigonometric terms, thus,

| | | | |
|---|---|---|---|
| Hyperbolic | sine | is expressed | - sinh |
| " | cosine | " " | - cosh |
| " | tangent | " " | - tanh |
| " | cotangent | " " | - coth |

A knowledge of the derivation and mathematical significance of hyperbolic functions is not required for the use of hyperbolic tables in evaluating the formulae given below.

The formulae for a and b for the equivalent network, shown in figure 24, for a length $\ell$ of uniform line having attenuation $\alpha$ and characteristic resistance $R_0$ are

*NOTE: The derivation of the formulae is given in chapter II of a book by A.E.Kennelly, entitled, "The Application of Hyperbolic Functions to Electrical Engineering Problems" (Publ. 1912, University of London Press).

**NOTE: For a short discussion of hyperbolic functions, see chapter I of a book by J.A. Fleming, entitled, "The Propagation of Electric Currents in Telephone and Telegraph Conductors" (Publ. 1915, D. Van Nostrand Co., New York). A more extensive discussion of hyperbolic functions and of their application to transmission problems is given in the book by A.E. Kennelly, entitled, "The Application of Hyperbolic Function to Electrical Engineering Problems."

$$a = R_o \tanh \frac{\ell\alpha}{2} \tag{107-a}$$

$$b = \frac{R_o}{\sinh \ell\alpha} \tag{108-a}$$

These may be written directly in terms of the series and shunt resistances $R_1$ and $R_2$, per unit length,

$$a = \sqrt{R_1 R_2}\ \tanh \frac{\ell}{2}\sqrt{\frac{R_1}{R_2}} \tag{107-b}$$

$$b = \frac{\sqrt{R_1 R_2}}{\sinh \ell \sqrt{\frac{R_1}{R_2}}} \tag{108-b}$$

A network having these values for the series and shunt arms can then be used to replace the uniform line of length $\ell$ in any circuit. It will be noted that the values of a and b vary with the length $\ell$ of the line which the equivalent network is used to represent.

4. Terminal Effects:

The value deduced above for the characteristic resistance of a uniform line is the resistance which would be obtained by a measurement across the terminals of an infinitely long uniform line. If the uniform line is short but has its far terminals closed through a resistance equal to $R_o$, the resistance across the near terminals will still be $R_o$. If, however, the far terminals of a short line are closed through some other resistance, some value other than $R_o$ will be obtained.

For a line such as figure 23, with the terminals 3-4 closed through a resistance $R_r$ the resistance across terminals 1-2

is*

$$R' = R_o \frac{R_r \operatorname{Cosh} l\alpha + R_o \operatorname{Sinh} l\alpha}{R_o \operatorname{Cosh} l\alpha + R_r \operatorname{Sinh} l\alpha} \qquad (109)$$

If the line is long, $l\alpha$ becomes large. For large values of $l\alpha$, cosh $l\alpha$ is practically equal to the sinh $l\alpha$ and R' becomes equal to $R_o$. That is, if a line is long so that its total attenuation is large, the effect of the resistance across the receiving terminals on the resistance across the sending terminals becomes negligible. The expression for R' is termed the sending end resistance of the line.

If the terminals at the far end are short circuited, then $R_r$ becomes zero and the sending end resistance $R_s$ becomes,

$$R_s = R_o \tanh l\alpha \qquad (110)$$

If the receiving end terminals are open, the sending end resistance, $R_f$, becomes

$$R_f = \frac{R_o}{\tanh l\alpha} \qquad (111)$$

From equations 110 and 111 it is seen that

$$R_o = \sqrt{R_s R_f} \qquad (112)$$

This offers a method of determining the value of $R_o$, if only a short piece of line is available. The characteristic resistance is the geometirc mean of the sending end resistances with the far end open and short circuited.

*NOTE: The derivation of formulae 109,110,111 and 114 is given in Chapter III sections 3 to 5 of the book by J.A.Fleming entitled, "The Propagation of Electric Currents in Telephone and Telegraph Conductors".

## 5. Transition Loss.

The transition loss at the terminals of a line, due to the connection to them of apparatus or circuits differing from the characteristic resistance of the line, is obtained from equation 50 by substituting for $R_1$ and $R_2$, in the formula, the values of $R_0$ and $R_r$, the characteristic resistance and the terminal resistance to which the line is connected. This expression assumes the form

$$\frac{2\sqrt{R_0 R_r}}{R_0 + R_r} \tag{113}$$

## 6. Transmission System.

Any transmission system for electrical power involves a generating element, a transmission line and a receiving element. In the transmission of direct currents the leakage resistance is usually so high as to be negligible in its effect on the transmission but for illustration it will be assumed here that the leakage is appreciable in reducing the current along the line.

A transmission system involving these three elements is shown in figure 25. The relations which have been derived for the current distribution in such a circuit will be applied to determine the efficiency of the system.

One way of treating such a system is to obtain the equivalent network of the uniform line (using equations 107 and 108) and determine the current distribution in the network of figure 26.

On the basis that the desirable result is to get the maximum power into the receiving element, the measure of the efficiency of the whole system is taken as the ratio of the power sent into $R_r$ in figure 26 to that which would be sent through the receiving

element if its resistance were equal to the internal resistance $R_G$ of the generating element and there were no transmission line between $R_G$ and $R_r$.

If it is desired to obtain the effect of the line above, the efficiency is expressed as the ratio of the current through $R_r$ with the transmission line in the system to the current through $R_r$ without the transmission line. The transition loss between $R_G$ and $R_r$ can then be expressed separately.

Referring to figure 26 the effect on the current through $R_r$ of inserting the transmission line is given by substituting the proper values in equation 63-c.

$$M_\ell = \frac{b\,(R_G + R_r)}{(R_G + a)(R_r + b + a) + b\,(a + R_r)} \qquad (114)$$

The effect of the transition loss is (refer equation 50)

$$M_t = \frac{2\sqrt{R_G\,R_r}}{R_G + R_r} \qquad (115)$$

The measure of the total efficiency of the system on the basis of equivalent current, compared to the best that could be done, assuming the generating element fixed, is then

$$= M_\ell\, M_t \qquad (116)$$

The ratio of the power expended in $R_r$ in the system of figure 26 to the power which could be delivered to it, if there were no transmission line, and $R_r$ were equal to $R_G$ is

$$= (M_\ell\, M_t)^2 \qquad (117)$$

The other method of determining the efficiency of the system does not involve the use of equivalent networks. From the constants $R_1$ and $R_2$ of the line the values of $R_o$ and $\alpha$ are determined. The resistance, R', across the terminals 1-2 is then evaluated from equation 109 for the sending end resistance of the line closed through a resistance $R_r$ at the far end. The current $I_1$ entering the line is then

$$I_1 = \frac{E}{R_G + R'} \tag{118}$$

The effect of the transition loss at the sending end on the current through $R_r$ is

$$M_{t_1} = \frac{2\sqrt{R_G\ R'}}{R_G + R'} \tag{119}$$

The effect of the attenuation in the line is

$$M_\alpha = \frac{I_2}{I_1} = e^{-l\alpha} \tag{120}$$

The resistance, R", across terminals 3-4, looking to the left is then given by equation 109 in which the line is closed at the far end through $R_G$. The measure of the transition loss at the receiving end is then

$$M_{t_2} = \frac{2\sqrt{R_r\ R''}}{R_r + R''} \tag{121}$$

The measure of the efficiency of the system on the basis of equivalent current is then

$$= M_{t_1}\ M_\alpha\ M_{t_2} \tag{122}$$

and on the basis of power is

$$= (M_{t_1} M_{\alpha} M_{t_2})^2 \qquad (123)$$

The results given by the two methods are of course the same. If the attenuation of the line is sufficiently great to make $R' = R'' = R_o$ the work involved in the second method is materially reduced since it is not necessary to evaluate $R'$ and $R''$.

The methods of determining the efficiency of the system of figure 25 are applicable to any circuit. If the sending and receiving terminals are multibranch networks, they can be reduced by means of Pollards theorem. If the transmission line consists of two dissimilar lines joined together as shown in figure 27 the equivalent network method can be used by deriving the network for each line separately and then combining the two networks into one. With the second method illustrated above, the portion of the system to the left of terminals 3-4 can be worked out in the same manner as for figure 25. Closing these terminals through the sending end resistance of the circuit to their right, then evaluating the impedance of the circuit to the left of 3-4, the transition loss at 3-4 can be determined and the current flow in the circuit to the right of 3-4 worked out.

## II - Alternating Currents

### A. General

In presenting the treatment of alternating current circuits it is assumed that the material given in Chapter XV of the book, "Elements of Electricity" by W.H.Timbie is known. In what follows some of the salient matters discussed in this chapter are repeated for the purpose of review and emphasis.*

The main characteristic of an alternating current is its periodic variation in magnitude and direction with time. Thus the current at any point in a circuit will increase from zero to a maximum value in one direction, then decrease to zero, increase to a maximum value in the opposite direction and then decrease to zero, repeating this cycle periodically. This variation in the magnitude of an alternating current, expressed in terms of the maximum value I is given by the following,

$$i = I \sin 2\pi f t$$

where i is the value of the current at any instant, t the time and f the frequency of the current, i.e., the number of cycles which occur per second.

This variation in the magnitude of an alternating current introduces certain factors which are not involved in the distribution of direct current in circuits.

#### 1. Effective Value

The magnitude of an alternating current is usually indicated by specifying its "effective value." The effective

*Note: for a more complete discussion of alternating currents refer to the following:"Electricity & Magnetism for Engineers, Part II, Electrostatics and Alternating Currents" by H.Pender (Publ.1919 by McGraw-Hill Book Co.,New York,Chapters XIV and XV.

value of an alternating current is defined as the magnitude of the direct current which when flowing in the same resistance as the alternating current generates in the same time the same heat as the alternating current. Since the heat generated is proportional to the square of the current, the effective value is equal to the square root of the sum of the squares of the various instantaneous values of the alternating current. For this reason, this is often referred to as the "root mean square" current. The effective value of an alternating current is

$$I_{eff} = \frac{1}{2}\sqrt{2}\ I_{max}.$$

That is, the effective current is equivalent to a direct current whose value is $\frac{1}{2}\sqrt{2}$ or approximately .707 times the maximum value which the alternating current reaches. Unless otherwise specified alternating currents are referred to by their effective values.

As mentioned above, the number of times which an alternating current passes through its cycle per second is referred to as the frequency of the current. The letter "f" is customarily used for the frequency and "p" for the value of 2 $\pi$ f.

In what follows it will be shown that the same laws which apply to direct currents apply also to alternating currents. Calculations in alternating current circuits can be made in the same way as in direct current circuits, it is only necessary to give a slightly different interpretation to the terms so as to take account of the two additional factors which are involved in alternating current problems, namely, the time relation between currents and voltages, and frequency.

2. Resistance.

In a circuit containing a source of alternating electromotive force of maximum value E and instantaneous value e = E sin 2 f t

in series with a resistance R, Ohm's law as stated for direct currents applies. It is:-

$$i = \frac{e}{R} \tag{124.}$$

$$I \sin 2\pi f\, t = \frac{E \sin 2\pi f\, t}{R} \tag{125.}$$

$$I_{max.} = \frac{E_{max}}{R} \tag{126.}$$

$$I_{eff.} = \frac{E_{eff.}}{R} \tag{127.}$$

That is, the ratio of voltage to current is the resistance for all ways of expressing the voltages and currents, instantaneous, maximum or effective. The a.c. voltage drop across the resistance has its maximum value at the same time that the current has its maximum value and so for all the corresponding values in the cycle. The voltage drop across a resistance and the current through the resistance are said to be "in phase."

The power which an alternating current dissipates in a resistance is

$$P = I_{eff.}\, E_{eff.} \tag{128.}$$

or as indicated above-

$$P = I_{eff}^{\,2}\, R \tag{129.}$$

3. Reactance.

Because of the fact that an alternating current is constantly varying, inductance and capacity in a circuit affect the flow of alternating current. The effect which these quantities have on the flow of the current is referred to as their reactance.

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The reactance $X_L$ offered by an inductance of L henrys to an alternating current of frequency f is

$$X_L = 2\pi f L \tag{130.}$$

The reactance $X_C$ offered by a condenser to the flow of a.c. is

$$X_C = \frac{1}{2\pi f C} \tag{131.}$$

It will be noted that the reactance of an inductance or an inductive reactance increases with frequency, while a capacity reactance decreases with frequency. In a circuit containing only a source of alternating electromotive force of voltage E and frequency f and an inductance L, the current which flows is

$$I = \frac{E}{2\pi f L} \tag{132.}$$

Similarly in a circuit containing only a capacity the current flowing is

$$I = \frac{E}{\frac{1}{2\pi f C}} = 2\pi f C E \tag{133.}$$

While a reactance can affect the current flow in a circuit, no energy can be dissipated in a reactance. Engrgy is stored in the electromagnetic and electrostatic fields associated with an inductance and capacity respectively, during one part of the cycle but the same amount of energy is released from these fields to the circuit in another part of the current cycle. Mathematically, the difference between the power which is released during one part of the cycle and that which is stored during the other part is expressed by giving the power which is released the positive sign and the power which is stored the negative sign. The sum of the two during this cycle is then zero.

[illegible]

It will be remembered that in the case of direct currents, power was defined as the product of current and voltage. This definition still holds when dealing with alternating currents for the instantaneous values. From this it follows that the power is positive when the instantaneous current and voltage have the same sign, and negative when they have opposite sign.

Figure 28 shows a plot for one cycle of the instantaneous current, voltage and power in an alternating current circuit having only inductive reactance. As pointed out the power is positive during half the cycle and negative during the other half. An inspection of the figure 28 shows that this requires a variation of the voltage and current such that when one of the two goes through its maximum value, the other is zero and vice versa. Successive maxima of the two, it will be noted, are one-quarter of a cycle apart. Such a time relation between current and voltage is referred to by saying that the two are a quarter of a cycle out of phase.

Figure 28 shows the relations in a circuit having only inductive reactance. The voltage drop across an inductive reactance reaches its positive maximum value when the current through the inductance is passing through zero towards its positive maximum, that is, the voltage variation is a quarter of a cycle ahead of the current variation. The current in this case is said to "lag" the voltage.

The voltage drop across a capacity reaches its maximum positive value when the current through the capacity is passing thru zero after decreasing from its maximum positive value, that is, the voltage variation is one quarter of a cycle behind the current variation. In this case the current is said to "lead" the voltage.

In combining reactances in a circuit, inductance and capacity reactance are taken to have different signs. It is usual to take the inductive reactance as positive and the capacity reactance as negative. Therefore, in adding an inductive and a capacity reactance in series the algebraic sum is taken to obtain the total reactance of the combination.

It is oftten convenient to plot alternating currents in polar coordinates as shown in figure 29. In this representation the lengths of the vectors represent their effective value, and one complete cycle is taken to be $360^{\circ}$. Fractions of a cycle are shown as corresponding fractions of this angle. A phase difference between current and voltage of one quarter cycle in this method of representation is shown by an angle of $90^{\circ}$ between the current and voltage vectors.

The vectors are assumed to rotate in a counter-clockwise direction. In figure 29, for instance, the current lags behind the voltage by $\theta^{\circ}$. Referring the phase of the current to that of the voltage, the current then is often written $I\backslash\!\overline{/\theta^{\circ}}$. On the other hand the voltage leads the current by $\theta^{\circ}$. Referring the phase of the voltage to the current, the voltage then is often written $E\underline{/\theta^{\circ}}$. In working out problems it is immaterial which vector is taken as reference so long as all phase angles are referred to the same vector throughout.

### 4. Impedance.

*The impedance of a circuit is defined as the ratio of the voltage across the circuit to the current flowing in the circuit.*

This statement corresponds to Ohm's Law for direct current circuits, but the term impedance replaces the term resistance. Taking in figure 29 the current I as the reference, it was pointed out that

the voltage can be written E $\angle\theta^{\circ}$. The impedance Z being the ratio of the voltage and the current then follows:

$$Z = \frac{E \angle\theta^{\circ}}{I} = \frac{E}{I} \angle\theta^{\circ}$$

Thus it will be seen that the impedance has an angle and can be regarded as a vector quantity.

It was pointed out above that in a circuit having only resistance, the current and voltage are in phase. A resistance, consequently, has zero phase angle. In a purely reactive circuit, however, it has been shown voltage and current are 90° out of phase. A reactance consequently has a 90° phase angle.

In plotting impedances as vectors it is customary to refer all angles to the horizontal axis. Figure 30 shows an impedance vector diagram. The resistance R is shown along the horizontal axis and the reactance X at an angle of 90° to this axis, or in the direction of the vertical axis. The impedance of the circuit is found by adding the resistance and the reactance geometrically, as shown in the figure. Thus the relation is obtained

$$Z = \sqrt{R^2 + X^2} \quad \angle\theta^{\circ} \qquad (134\text{-a})$$

where θ is determined by the relation

$$\tan\theta = \frac{X}{R}$$

As pointed out above an inductive reactance is taken as positive and a capacity reactance as negative. Thus the combination of a resistance R, inductive reactance $X_L$ and capacity reactance $X_C$ is represented by an impedance

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \angle\theta^{\circ} \qquad (134\text{-b})$$

5. Vector Notation:*

In referring to impedances in alternating current circuits it is customary to use a Vector notation. The letter "j" is used in the notation to indicate a phase difference of 90 degrees. Thus the expression

$$Z = R + j\,X \qquad (135)$$

indicates that the reactance X is 90 degrees out of phase with R in its effect on the current, and the total impedance Z of such a combination is obtained by adding R and X as vectors with this difference of direction of 90 degrees. Similarly, equation (134-b) can be expressed,

$$Z = R + j\,X_L - j\,X_C \qquad (136\text{-a})$$

$$= R + j\,(X_L - X_C) \qquad (136\text{-b})$$

If two impedances

$$Z_1 = R_1 + j\,X_1 \qquad (137)$$

$$\text{and } Z_2 = R_2 + j\,X_2 \qquad (138)$$

are in series and it is desired to obtain the impedance of the combination, the equations can be added as

$$Z_1 + Z_2 = R_1 + R_2 + j\,(X_1 + X_2) \qquad (139)$$

The above expressions may also be subtracted, multiplied and divided. In multiplying two factors prefixed by j, as $j\,X_1$ and $j\,X_2$, the product is $j^2\,X_1\,X_2$. If j indicates a phase shift of 90 degrees, then a vector shifted 90 degrees twice, has changed its direction by

*Note: For a more complete discussion of Vector notation refer to the following:-
"Alternating Currents and Alternating Current Machinery" by D.C.&J.P.Jackson (Publ.1913 by the MacMillan Co.,New York) Chapter IV, Sections 62 and 63, and Chapter V; and "The Propagation of Electric Currents in Telephone and Telegraph Conductors," by J.A.Fleming,Chapter I, or "Electricity and Magnetism for Engineers, Part II, Electrostatics and Alternating Currents," by H. Pender, Chapter XVII.

180 degrees, i.e. it points opposite to its previous direction. In mathematical terms

$$j^2 X_1 X_2 = -X_1 X_2 \tag{140}$$

From this, $j^2 = -1$ (141)

This valuation of "j" gives a means of determining the effect of powers of "j". Thus,

$$j^2 = -1 \tag{142}$$

$$j^3 = j\,(-1) = -j \tag{143}$$

$$j^4 = 1 \tag{144}$$

$$j^5 = j \tag{145}$$

and so on.

Also

$$\frac{1}{j} = \frac{j}{-1} = -j \tag{146}$$

For an impedance

$$Z = R+jX,$$

it has been noted above that Z is a vector in which the magnitude of Z represents the length, and the angle at which it is placed, with respect to the vector representing R, is the angle $\theta$ whose tangent is $\frac{X}{R}$

Thus $R+jX = Z\underline{/\theta}$ (147)

$$Z\underline{/\theta} = Z \cos \theta + j\,Z \sin \theta \tag{148}$$

$$R = Z \cos \theta \tag{149}$$

$$X = Z \sin \theta \tag{150}$$

The expression for a current in terms of the voltage E across a circuit and the impedance $Z\underline{/\theta}$ of the circuit is

$$I = \frac{E}{Z\underline{/\theta}} = \frac{E}{R+jX} \tag{151}$$

Multiplying both numerator and denominator by R - jX gives

$$I = \frac{E(R-jX)}{R^2-j^2X^2} = E\frac{R}{R^2+X^2} - jE\frac{X}{R^2+X^2} \quad (152)$$

Now $R^2 + X^2$ is the square of the magnitude of Z

$$I = \frac{ER}{Z^2} - j\frac{EX}{Z^2} \quad (153)$$

$$= \frac{E}{Z}\left(\frac{R}{Z} - j\frac{X}{Z}\right) \quad (154)$$

Referring to figure 30

$$\frac{R}{Z} = \cos\theta \quad (155)$$

$$\frac{X}{Z} = \sin\theta \quad (156)$$

Therefore $I = \frac{E}{Z}(\cos\theta - j\sin\theta)$ (157)

or $I = \frac{E\,\overline{/\theta}}{Z}$ (158)

It has been shown (see equation 147) that a vector may be written in two forms, either in the form a+jb or in the form $c\underline{/\theta^o}$

In general when adding or subtracting vectors it is convenient to use the first form, whereas when multiplying or dividing vectors the second form is convenient.

## 6. Combination of Impedances.

With this vector notation, impedance elements can be combined in the same way as resistance elements in series or parallel, provided the "j" is properly taken into account. Terms not having the symbol "j" can be added directly and likewise terms having the symbol j. Terms not having the symbol j cannot be added to terms prefixed with the symbol j. Terms can be multiplied or divided by each other irrespective of whether or not they have the symbol j prefixed, but the symbol if associated

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with either term must be taken into account as indicated above.

Given a resistance $= R$,

an inductive reactance $= +jX$,

a capacity reactance $= -jX$.

These terms when used to indicate the impedance elements in a circuit, can be combined in the same way as resistances for direct current circuits. Elements in series can be added to obtain the combined effect, thus,

$$Z = R_1+R_2+jX_1-jX_2 \tag{159}$$

Likewise, the impedance of a combination of two impedance elements in parallel is equal to the ratio of their product to their sum. Thus the impedance of R and jX in parallel is

$$Z = \frac{jRX}{R+jX} \tag{160}$$

Similarly for $jX_1$ and $-jX_2$ in parallel

$$Z = \frac{-j\,X_1\;j\,X_2}{j\,X_1-j\,X_2} = \frac{-j^2\,X_1X_2}{j(X_1-X_2)} = \frac{-j\,X_1X_2}{X_1-X_2} \tag{161}$$

For two complex impedances $R_1 + j\,X_1$ and $R_2-j\,X_2$ in parallel the impedance of the combination is

$$Z = \frac{(R_1+jX_1)\ \ (R_2-jX_2)}{(R_1+jX_1)+(R_2-jX_2)} \tag{162-a}$$

This may be reduced to

$$Z = \frac{R_1\,R_2+jX_1R_2-jX_2R_1-j^2X_1X_2}{R_1+R_2+jX_1-jX_2} \tag{162-b}$$

$$Z = \frac{R_1R_2+X_1X_2+j(X_1R_2-X_2R_1)}{R_1+R_2+j(X_1-X_2)} \qquad (162\text{-c})$$

The term admittance, Y, is used to designate the reciprocal of impedance.

$$Y = \frac{1}{Z} \qquad (163)$$

The real part of the admittance, G, is called the conductance and the imaginary part, B, the susceptance. Between the admittance, conductance and susceptance the relation exists that

$$Y = G + jB \qquad (164)$$

Referring to equation (153) it will be seen that the conductance G and susceptance B of a circuit having an impedance, $Z = R + jX$, are

$$G = \frac{R}{Z^2} \qquad (165)$$

$$B = \frac{-X}{Z^2} \qquad (166)$$

In a circuit consisting of two parallel branches, one of which has an admittance, $Y_1 = G + jB_1$, and the other of which has an admittance, $Y_2 = G_2 + jB_2$, the total admittance, Y, of the combination is the vector sum of the two admittances $Y_1$ and $Y_2$.

$$Y = Y_1 + Y_2 \qquad (167)$$

$$= G_1 + G_2 + j(B_1 + B_2) \qquad (168)$$

This relation can be stated generally that --

<u>When several impedances are in parallel, the total admittance of the combination is the vector sum of the separate admittances.</u>

<u>7. Resonance.</u>

In an a.c. circuit having both capacity and inductance in series the reactance is zero whenever the capacity reactance is equal

in magnitude to the inductive reactance, i.e. when $X_L = X_C$ as can be seen from equation 136-b.

From equation 130 and 131 it follows for this condition that

$$2\pi f L = \frac{1}{2\pi f C} \tag{169}$$

or $$f = \frac{1}{\sqrt{4\pi^2 L C}} = \frac{1}{2\pi\sqrt{LC}} \tag{170}$$

This shows that -

<u>In a circuit having capacity and inductance in series there is one frequency at which the total reactance is zero. This frequency is called the resonant frequency of the circuit.</u>

There are also circuits which have several resonant frequencies at which their reactance is zero.

When the resistance of a circuit is small relative to the capacity reactance and the inductive reactance, the impedance at the resonant frequency assumes a very small value. Very large currents may then be set up in the circuit compared with those set up at other frequencies. In all circuits having capacity and inductance in series, it is therefore necessary to know the resonant frequency (or frequencies, if there are several) and to determine the resonant effects, as these may under certain conditions be very pronounced and considerably affect the behavior of the circuit at that frequency.

A similar relation exists when capacity and inductance are in parallel in a circuit. The expression for two reactances $X_1$ and $X_2$ in parallel is

$$X = \frac{X_1 X_2}{X_1 + X_2}$$

Let $X_1$ represent an inductive reactance and $X_2$ a capacity reactance, then X becomes infinite when $X_1 = X_2$. The frequency at which this occurs is called the anti-resonant frequency of the circuit. This frequency is given by equation 170, where L and C are the inductance and the capacity in parallel.

B - A.C. Networks*

As has been indicated several times, the laws and relations which have been discussed and derived for the flow of direct currents in circuits all apply to alternating currents, provided that the term resistance in direct current circuits is generalized to impedance in alternating current circuits and that currents, voltages and impedances are treated as vectors, which have phase relations as well as magnitudes.

Thus Ohm's Law for a.c. circuits may be stated:-

In a circuit containing a source of alternating electromotive force, the current which flows is the quotient of the voltage divided by the impedance of the circuit. Also the voltage drop across any part of a circuit is equal to the product of the current and the impedance of that part of the circuit.

Kirchoff's first law for a.c. circuits is

The vector sum of the currents flowing to any point in a network is zero.

Kirchoff's second law for a.c. circuits is

In any closed electric circuit the vector sum of the electromotive forces and potential drops is zero.

Consider the circuit of figure 31 which is similar to that of figure 3 except that E is an a.c. voltage and the elements of the circuits are impedances. The formulae for the currents in the circuit of figure 31 are analogous to those derived for the circuit of figure 3.

*NOTE: A simple discussion on the solution of networks is given in Chapter VI of "Alternating Currents and Alternating Current Machinery", by D.C. and J.P. Jackson.

For mesh a d e

$$E = I_1 Z_1 + I_3 Z_3 \tag{171}$$

and for mesh a b c d

$$I_2 Z_2 = I_3 Z_3 \tag{172}$$

and using Kirchoff's first law

$$I_1 = I_2 + I_3 \tag{173}$$

In these equations the quantities are of course all vectors. Solving these equations in the same way as for the circuit of figure 3 the current equations are

$$I_1 = \frac{E}{Z_1 + \dfrac{Z_2 Z_3}{Z_2 + Z_3}} \tag{174}$$

$$I_2 = \frac{Z_3}{Z_2 + Z_3} I_1 \tag{175}$$

$$I_3 = \frac{Z_2}{Z_2 + Z_3} I_1 \tag{176}$$

With the R+jX values of the impedances substituted in these questions, vector values of the currents are obtained.

The other networks discussed under direct currents can be similarly treated. The effect of the introduction of series and shunt impedances in networks as derived for d.c. can also be generalized to cover a.c. circuits.

It should be noted, however, that power can be dissipated only in resistances and not in reactances and that if a current $I_2$ is delivered to a receiving element having an impedance $Z_r = R_r + jX_r$ the power delivered to this element is not $I_2{}^2 Z_r$ but $I_2^2 R_r$.

## C - TRANSFORMERS

A transformer is a device for alternating currents, which has no analogy in the direct current circuit because its operation depends upon the fact that an alternating current is always varying in magnitude. A transformer consists essentially of two coils associated so that the magnetic field set up by a current through one coil passes through the other coil. If a voltage is impressed across the terminals of the first coil, any variation in the current sent through the coil sets up a corresponding variation in the electromagnetic field which this coil produces. The variations of the field through the second coil in turn induce a potential in this second coil which is proportional to the voltage across the first coil. To make the linking of the two coils by the same magnetic field, the two windings of a transformer are usually wound, one over the other around an iron core, and the iron core is usually a closed core to make the magnetic circuit efficient.

### 1. Perfect Transformer.

Assume two turns of wire to be placed on one iron core so that all flux set up in one will link with the other. A varying voltage which is impressed across the terminals of one will induce in the other a voltage of magnitude as the applied voltage and of opposite direction. If three turns are placed on the iron core and a voltage is applied to one of them, voltages of the same magnitude as the applied voltage are induced in each of the other two. Connecting these latter turns then in series so that the voltages in

them are added, double the applied voltage is obtained across the combination. Quite generally, assume that two windings are placed on one core so that all the flux set up in one will link with the other, the one winding having $N_1$ turns and the other $N_2$ turns. A varying voltage $V_1$ which is impressed across the outer terminals of the first winding then produces $\frac{V_1}{N_1}$ volts across each turn of this winding, as the voltage divides equally between the turns provided they are all alike. A voltage of the same magnitude $\frac{V_1}{N_1}$ is induced in each turn of the second winding. As the second winding has $N_2$ turns which are all connected series aiding, the voltage $V_2$ measured across the outer terminals of the second winding will be $\frac{V_1}{N_1} N_2$. In dealing with transformers it is customary to refer to the two windings of turns $N_1$ and $N_2$, respectively, as the primary and the secondary windings. The above relation between the voltages $V_1$ and $V_2$ can be written as follows:

$$\frac{V_1}{V_2} = \frac{N_1}{N_2} \tag{177a}$$

or

$$V_1 N_2 = V_2 N_1 \tag{177b}$$

A transformer presents a means of transferring power from a circuit connected to one winding to a second circuit connected to the other winding without any direct connection between the two circuits. Figure 32 shows a transformer connecting a generator of

voltage E and internal impedance $Z_G$ to an impedance $Z_2$. Power is delivered to the transformer which in turn delivers the power to the load $Z_2$. For a perfect transformer the power delivered by the transformer from winding $N_2$ would be equal to the power put into winding $N_1$, that is, the transformer would be 100 per cent. efficient. Such a transformer consequently would introduce no losses, if placed in a circuit. Since it is practicable to make commercial transformers which approach this efficiency very closely, this discussion will be confined for a time to such a transformer.

Since no power is lost in the transformer and since the phase relation between the voltage and current on the two sides of the transformer is exactly the same, the product of the voltage $V_1$ across the primary by the current $I_1$ through the primary winding is equal to the corresponding product for the secondary winding, that is

$$V_1 I_1 = V_2 I_2 \tag{178}$$

or
$$\frac{V_1}{V_2} = \frac{I_2}{I_1} \tag{179}$$

Substituting the value of $\frac{V_1}{V_2}$ from equation (177a)

$$\frac{I_2}{I_1} = \frac{N_1}{N_2} \tag{180}$$

The relations in equations (180) and (177) can be expressed as follows:

The currents through the two windings of a transformer are inversely proportional to the number of turns in the two windings, and the voltages across the two windings are directly proportional to the number of turns in the two windings.

Now

$$V_2 = I_2 \; Z_2 \tag{181}$$

Substituting the value of $V_2$ from equation (177) and the value of $I_2$ from equation (180)

$$V_1 \; \frac{N_2}{N_1} = \frac{I_1 \; N_1}{N_2} \; Z_2 \tag{182a}$$

Then

$$\frac{V_1}{I_1} = Z_1 = \left(\frac{N_1}{N_2}\right)^2 Z_2 \tag{182b}$$

This relation can be stated that -

The impedance across the terminals of the primary winding of a transformer when the secondary winding is closed through an impedance $Z_2$ is equal to the square of the ratio of the number of turns in the two windings times $Z_2$.

The current through the first winding is then

$$I_1 = \frac{E}{Z_G + Z_1} = \frac{E}{Z_G + \left(\frac{N_1}{N_2}\right)^2 Z_2} \tag{183}$$

If $N_1$ is greater than $N_2$ the impedance which is connected to the generator terminals is greater than $Z_2$ by the factor $\left(\frac{N_1}{N_2}\right)^2$. Correspondingly the current through $Z_2$ is greater than that put

out by the generator by the factor $\frac{N_1}{N_2}$

If $N_1 = N_2$, then $Z_1 = Z_2$ and $I_1 = I_2$ and $V_1 = V_2$.

$$\text{Therefore } I_1 = \frac{E}{Z_G + Z_1} = \frac{E}{Z_G + Z_2} \qquad (184)$$

The above shows that placing a unity ratio transformer in the circuit does not change the current through $Z_2$, but breaks the circuit continuity between the generator and $Z_2$. A unity ratio transformer consequently discriminates against the passing of direct current between the circuits on either side of the transformer but still allows alternating currents to be transferred without appreciable loss.

As has been indicated, the transformer provides a means of changing the magnitudes of alternating voltages and currents and also of modifying the magnitude of the impedance of any circuit or apparatus to which it is connected. One application which is useful in the telephone system is the placing of a transformer between two impedances which are unequal in order to reduce the transition loss at the point. Thus in Figure 32, the ratio of the turns in the transformer windings can be adjusted to make $\left(\frac{N_1}{N_2}\right)^2 Z_2$ equal in magnitude to $Z_G$. The transformer cannot, however, change the phase of the impedance.

## 2. Physical Transformer of Unity Ratio.

In dealing with circuits in which a transformer is connected, it is often convenient to use an equivalent T-network to replace the transformer. The design of such a network is here given for the case of a unity ratio transformer. If the impedance of one winding of the transformer (refer to Figure 33), say the primary, is $Z_p$ when the other, or secondary, winding is open, and the impedance of the secondary winding is $Z_S$ with the primary open-circuited, the impedances a, b and c of the equivalent T-network are given by expressions corresponding to equations (77), (78), and(79). Remembering that the impedance of the primary with the secondary short-circuited is zero by equation (182b), it is

$$c = \sqrt{Z_S Z_p} \tag{185}$$

a and b have the values

$$a = Z_p - c \tag{186}$$

$$b = Z_S - c \tag{187}$$

The equivalent network of the transformer is shown in Figure 34. It will be noted from an inspection of this figure that with the secondary open, the voltage across the terminals of the secondary $V_{2_o}$ for a current $I_1$ in the primary is equal to the voltage drop in the shunt impedance c, i.e. $I_1 c = V_{2_o}$ or $c = \frac{V_{2_o}}{I_1}$. This ratio $\frac{V_{2_o}}{I_1}$ is called $Z_M$, the mutual impedance of the transformer

$$Z_M = \frac{V_{2_o}}{I_1} \tag{188}$$

Similarly, $Z_S$ and $Z_p$ are referred to as the self impedances of the transformer.

Thus the result is obtained that

$$c = Z_M \tag{189}$$

From equation (185) it then follows that

$$Z_M = \sqrt{Z_S\ Z_p} \tag{190}$$

If the transformer is unity ratio, as assumed here, that is if $N_1 = N_2$, then

$$Z_p = Z_S \tag{191}$$

$$\text{and} \qquad Z_M = \sqrt{Z_p\ Z_S} = Z_p = Z_S \tag{192}$$

In this case the equivalent network of the transformer reduces to a shunt impedance of magnitude $Z_M$.

Above, the perfect transformer was defined as introducing no loss when placed in a circuit. The equivalent network shows the mutual impedance of the transformer as a shunt across the line. In the discussion of networks it was shown that the loss introduced by a shunt impedance is the smaller, the larger the line impedance. In order to introduce no loss when connected between resistances, a shunt impedance would have to be indefinitely large, i.e. a perfect transformer has infinite mutual impedance.

In the physical transformer, of course, all impedances are finite in magnitude. In other words the transformer takes a small open circuit current. This corresponds to the current required to set up the flux in the core.

In addition, transformers cause a loss in a circuit due

to the fact that the windings have resistance and dissipate heat when current flows through them. Also all the flux produced by one winding does not pass through all of the other winding, some of the flux "leaking" away from the core path. Furthermore, a small amount of loss is occasioned by currents induced in the core itself, (eddy current losses) and in the setting up of the necessary flux in the core (hysteresis losses).

For these reasons, in a unity ratio transformer, the mutual impedance is not equal to the primary and secondary impedances but is less than they are by a factor which is termed the leakage impedance of the primary or secondary. This leakage impedance includes the resistance of the windings and the reactance caused by the leakage flux. The mutual impedance has a resistance component corresponding to the losses in the core.

3. Physical Transformer of Inequality Ratio.

In determining the equivalent network for an inequality ratio transformer, the same formulae apply as for a unity ratio transformer. It will be noted that the series impedances of the T network become the larger, the larger the voltage ratio of the transformer, and that the sign of the series impedance on the high voltage side has its sign reversed. In the circuit of Figure 35 the current on the primary side of the transformer is

$$I_1 = \frac{E}{Z_G + Z_p - \frac{Z_M^2}{Z_2 + Z_S}} \qquad (193)$$

The current on the secondary side of the transformer is related to that on the primary side by the equation $I_2 = I_1 \frac{Z_M}{Z_S + Z_2}$.

D. - Uniform Lines

The transmission line for alternating currents possesses four properties, which are termed "primary constants". These are the series resistance of the wires, the inductance of the circuit, the capacity between the two sides of the circuit, and the leakage resistance, or conductance, between the two sides of the circuit. These quantities are customarily designated respectively by R, L, C and G. Of these, R and L are series constants and C and G are shunt constants. A short piece of such a line may be represented by the circuit shown in Figure 36 where these values per unit length are shown lumped. Actually, of course, they are distributed.

The series impedance element, $Z_1$, of such a line for a frequency f is

$$Z_1 = R + j2\pi fL = R + jpL \qquad (194)$$

The shunt impedance element, $Z_2$ may be obtained by adding together the admittances of each of the two and taking the reciprocal.

Thus $$Z_2 = \frac{1}{G + j2\pi fC} = \frac{1}{G + jpC} \qquad (195)$$

1. Characteristic Impedance

A uniform a.c. line has a characteristic impedance just as a uniform direct current line composed of resistances only, has a characteristic resistance. For the resistance line, the characteristic resistance was

$$R_0 = \sqrt{R_1 R_2} \qquad (99\text{-a})$$

Thus, by analogy the characteristic impedance of the a. c. line is

$$Z_0 = \sqrt{Z_1 Z_2} = \sqrt{\frac{R + jpL}{G + jpC}} \qquad (196)$$

It will be noted that, since the factor p involves the frequency of the current, the impedance will change with frequency. If this frequency becomes zero then

$$Z_o = \sqrt{\frac{R}{G}} \qquad (197)$$

which is equivalent to equation 99-b when $R_1$ is the series resistance of the line and $G_2$ the leakage conductance.

The variation of the impedance of a No. 8 B. W. gauge open wire line with frequency is shown by figure 37. This line has the following constants per mile length of circuit

| | | |
|---|---|---|
| R = | 4.14 | ohms |
| C = | .00914 | microfarads |
| L = | .00337 | henrys |
| G = | 0 | mhos |

The impedance is shown in both forms R+jX and $Z/\underline{\theta}$.

Figure 38 shows the characteristic impedance of a No. 19 B & S gauge cable circuit which has the following constants.

| | | |
|---|---|---|
| R = | 88 | ohms |
| C = | .054 | microfarads |

In cable circuits the values of L and G are small, and have been neglected here.

2. Propagation Constant.*

When an alternating current is passed over a given length of line having resistance, inductance, capacity and conductance it is not only attenuated, that is, reduced in its effective value, but the instantaneous values of the currents at the two ends of the line are

* Note: For a more extensive presentation of the matter under this heading refer to a book by J. A. Fleming "The Propagation of Electric Currents in Telephone and Telegraph Conductors" Chapter III.

not necessarily at the same relative positions in the cycle. The factor which takes both of these changes into account is termed the "propagation constant." Again using the analogy to the resistance line where the attenuation was found to be

$$\alpha = \sqrt{\frac{R_1}{R_2}} \tag{103}$$

The propagation constant, $\gamma$, is

$$\gamma = \sqrt{\frac{Z_1}{Z_2}} = \sqrt{\frac{R+jpL}{\frac{1}{G+jpC}}} \tag{198-a}$$

$$= \sqrt{(R+jpL)\ (G+jpC)} \tag{198-b}$$

This will be seen to have a value which will include terms with and without j. The propagation constant may then be expresses

$$\gamma = \alpha + j\beta \tag{199}$$

where $\alpha$ is the "attenuation constant" and $\beta$ is termed the "wave length constant." $\alpha$ and $\beta$ can be determined separately from equation 198-b to have the following values.

$$\alpha = \sqrt{\frac{1}{2}\left\{\sqrt{(R^2+p^2L^2)\ (G^2+p^2C^2)} + (GR-p^2LC)\right\}} \tag{200}$$

$$\beta = \sqrt{\frac{1}{2}\left\{\sqrt{(R^2+p^2L^2)\ (G^2+p^2C^2)} - (GR-p^2LC)\right\}} \tag{201}$$

These expressions for $\alpha$ and $\beta$ appear rather formidable and it is usually easier to evaluate them by working out equation 198-b. The term without j is then the value of $\alpha$ and the term with j the value of $\beta$.

If the constants used in the formula for the propagation constant

are the values per mile length of circuit, the equation gives the propagation constant per mile. The pregation constant for any length "$\ell$" miles will be "$\ell$" times the value per mile.

The ratio of the currents at the two ends of a line of length $\ell$, where $I_1$, is the current at the sending end and $I_2$ at the receiving end, is

$$\frac{I_2}{I_1} = e^{-\gamma\ell} = e^{-(\alpha+j\beta)\ell} \qquad (202)$$

This can be written

$$\frac{I_2}{I_1} = e^{-\alpha\ell} \underline{/\beta\ell} \qquad (203)$$

This means that the magnitude of

$$I_2 = I_1 e^{-\alpha\ell} \qquad (204)$$

and that the two currents differ in phase by the angle $\beta l$, where $\beta$ is expressed in radians, a radian being $\frac{360 \text{ degrees}}{2\pi} = 57.3$ degrees.

Figure 39 shows for a given instant a plot of the magnitude and direction of the current at the various points along a No. 8 B. W. gauge open wire line for a frequency of 800 cycles. Figure 40 gives a picture of the currents, one quarter cycle later, and figure 41 a half cycle later than is indicated on the first drawing.

The envelopes of these curves show how the current is attenuated along the line. These curves show the attenuation of the maximum values of the current wave, but the attenuation of the effective value of the current would, of course, be directly proportional to this at all points.

Referring now to figure 39, it will be seen that at the instant that this picture of the current along the line is taken, the current at a point 113 miles out is flowing in the opposite direction to the current entering the line. At points 56.5 miles

and 169.5 miles out the instantaneous currents are zero. At a point 226 miles out, the current is a maximum flowing in the same direction as the input current to the line. Between the point a and point d the current has shifted through a complete cycle. This length a d is referred to as the wave length of this line for an 800 cycle current since a complete wave or cycle is included in the length.

This wave length, $\lambda$, may be evaluated from the equation

$$\lambda = \frac{2\pi}{\beta} \qquad (205)$$

Since $\beta$ is the phase shift per mile of line, the length required to shift a complete cycle will be the quotient obtained by dividing 360 degrees (equals $2\pi$), which is a cycle, by the shift $\beta$ per mile.

Since a wave length is one cycle in length, the velocity of propagation of the current along the line is the product of the length of a cycle by the number of cycles per second, i.e. the frequency.

The velocity W then is,

$$W = f\lambda \qquad (206)$$

since

$$\lambda = \frac{2\pi}{\beta} \qquad (205)$$

$$W = \frac{2\pi f}{\beta} = \frac{p}{\beta} \qquad (207)$$

From the formula for the velocity of propagation, it can be seen that this quantity varies with frequency.

It can be shown that if the constants of the line are related so that

$$\frac{R}{L} = \frac{G}{C} \qquad (208)$$

or $RC = LG$ (209)

the attenuation constant becomes

$$\alpha = \sqrt{RG} \tag{210}$$

$$\beta = p\sqrt{CL} \tag{211}$$

hence

$$W = \frac{1}{\sqrt{CL}} \tag{212}$$

For this case it is seen that the attenuation is independent of the frequency and hence is the same for all frequencies, likewise, the velocity of propagation is the same for all frequencies. A line having this relation of its fundamental constants is said to be distortionless in that it transmits all frequencies equally well. Reference will be made to this relation later in connection with the telephone transmission line.

For many of the cables used for telephone transmission such as the No. 19-gauge cable for which the constants were given on page 68, the inductance and conductance are so small that they can be neglected in their effect on the propagation. If it is assumed that L and G are zero, equations 200 and 201 for $\alpha$ and $\beta$ reduce to

$$\alpha = \sqrt{\frac{pRC}{2}} \tag{213}$$

$$\beta = \sqrt{\frac{pRC}{2}} \tag{214}$$

The attenuation and the wave length constant are equal for this case and both vary as the square root of the frequency.

3. Reflection Loss.

If a line having a characteristic impedance of $Z_0 \underline{/\theta}$ is connected to terminal apparatus having an impedance $Z_r \underline{/\phi}$, the current flowing into the terminal apparatus is different from what

it would be if the terminal apparatus had the same impedance as the line. This effect on the received current of bringing together two different impedances is called "reflection loss". Referring to equation 50, which was derived to give the effect of connecting together two unequal resistances and substituting vectus impedances therein, the magnitude of the reflection loss at the junction of two impedances is indicated by the expression

$$= \frac{2\sqrt{Z_o \angle\theta \; Z_r \angle\phi}}{Z_o \angle\theta + Z_r \angle\phi} \qquad (215)$$

4. Transition Loss.

In deriving the expression for the transition loss between two resistances it was stated that it was the loss as compared to the condition of delivering the maximum power to the receiving element. Consider now the circuit of Figure 42 where $Z_1$ is the impedance of a source of a.c. and $Z_2$ is the impedance of a device to which a.c. power is to be delivered. The current delivered to $Z_2$ is

$$I = \frac{E}{Z_1 + Z_2} \qquad (216\text{-a})$$

$$= \frac{E}{R_1 + R_2 + j\,(X_1 + X_2)} \qquad (216\text{-b})$$

The power $P_2'$ delivered to $Z_2$ is the square of I times the resistance component of $Z_2$, that is

$$P_2' = \left(\frac{E}{R_1 + R_2 + j\,(X_1 + X_2)}\right)^2 R_2 \qquad (217)$$

For $R_1$ and $R_2$ fixed, the value of $P_2'$ increases as $X_1 + X_2$ becomes smaller.

The limiting case, $X_1 + X_2 = 0$, is obtained when $X_2 = -X_1$. If this adjustment can be made the resistances alone remain, and it has already been shown that for this case the maximum power is delivered to $R_2$

The equivalent current ratio is the square root of these relations or

$$= \frac{2\sqrt{R_1R_2}}{R_1+R_2+j(X_1+X_2)} \tag{221-b}$$

$$= \frac{2\sqrt{Z_1Z_2 \cos\theta \cos\phi}}{Z_1 \angle\theta + Z_2 \angle\phi} \tag{221-c}$$

The difference between reflection and transition losses will be apparent when the difference in the two reference conditions is remembered. The reflection loss is referred to the condition that the two impedances at the junction of which the loss occurs are equal in magnitude and phase. The transition loss is referred to the condition that the two impedances are "conjugate", i.e. equal in magnitude but of opposite phase. The latter, as has just been shown, is the condition for maximum power transfer. Thus having two impedances in a circuit equal is not the condition for maximum power transfer. It is possible to design the circuit more efficiently by introducing a phase difference between the two impedances. From this it follows that it is possible to have a reflection "gain" at the junction of two impedances, whereas it is never possible to have a transition gain.

In many cases the impedance of terminal apparatus cannot be changed either in magnitude or phase to give the maximum power transfer. In such cases a transformer can be used to make the magnitudes of the two impedances equal. The loss which could be eliminated between the two impedances of figure 42, for instance, by inserting a transformer between the two impedances, is indicated by the equivalent current ratio

$$\frac{2\sqrt{Z_1 Z_2}\ \cos\left(\frac{\theta-\phi}{2}\right)}{Z_1\angle\theta + Z_2\angle\phi} \tag{222}$$

This is on the basis that the transformer itself causes no loss.

Returning to the case where $Z_1$ and $Z_2$ are unequal, it is possible to place a transformer having a turn ratio of n between the impedances so that

$$R_1' = n^2 R_2 \tag{223}$$

By placing a compensating reactance $X_3$ in series with $X_2$, $X_1$ can be made

$$X_1 = -n^2(X_2+X_3) \tag{224}$$

This presents the conjugate impedance to $Z_1$, and so maximum power is delivered to $R_2$.

When it is not possible to insert adjusting reactances, the best transformer to use is one which has the turn ratio

$$n^2 = \frac{Z_1}{Z_2} \tag{225}$$

5. Equivalent Network.

The equivalent network of a length of uniform line, for alternating currents, can be obtained by the same formulae used for the resistance line, provided the characteristic impedance $Z_0$ be substituted for $R_0$ and the propagation constant for the attenuation constant in equations 107-a and 108-a

$$a = Z_0 \tanh \frac{l\gamma}{2} \tag{226}$$

$$b = \frac{Z_0}{\sinh l\gamma} \tag{227}$$

a is the value of each of the two series arms and b the value of the shunt arm of the equivalent network.

6. Terminal Effects:

The formulae for the sending end impedance for a line of length l having a characteristic impedance $Z_o$ and propagation constant $\gamma$ with the receiving end (1) closed through an impedance $Z_r$, (2) short-circuited and (3) open circuited are readily obtained from the corresponding expressions for the d.c. line (equations 109, 110 and 111).

The sending end impedance with the far end closed through an impedance $Z_r$ is:

$$Z' = Z_o \frac{Z_r \cosh \gamma l + Z_o \sinh \gamma l}{Z_o \cosh \gamma l + Z_r \sinh \gamma l} \tag{228}$$

The sending end impedance with the far end short-circuited is:

$$Z_S = Z_o \tanh \gamma l \tag{229}$$

The sending end impedance with the far end open circuited is:

$$Z_f = \frac{Z_o}{\tanh \gamma l} \tag{230}$$

7. Transmission System:

To determine the operation of an a.c. transmission circuit consisting of a source of a.c. of impedance $Z_g$, a uniform line of length $l$, characteristic impedance $Z_o$, propagation constant $\gamma$, connected to a receiving element of impedance $Z_r$, the procedure is similar to that outlined for the direct current circuit containing only resistances. The formulae correspond in the two cases, the only difference being that the a.c. quantities are all vectors and their phase relations must be taken into account.

In connecting a piece of apparatus in a circuit to the end of a line, maximum efficiency, as was pointed out, is obtained if the two impedances connected together are conjugate. If this is

not the case they can be made so by associating with one of them a compensating reactance and a transformer. If it is not possible to adjust the reactances, the best thing is to insert a transformer which will make the two impedances equal in magnitude.

## E- LOADING

### 1. General

In a uniform line which is carrying an alternating current, it has been shown that as the current passes along the line it is attenuated or decreased in amount. If $I_1$ is the current at one point, the magnitude of the current $I_2$ at some point l units of length further along the line is represented by the following equation.

$$I_2 = I_1 e^{-l\alpha} \qquad (231)$$

where $\alpha$ is the attenuation constant of the line. The relation between the voltages at the two points is given by a corresponding equation to be

$$V_2 = V_1 e^{-l\alpha} \qquad (232)$$

If the characteristic impedance of the line is $Z_0 \underline{/\theta}$ then

$$\frac{V_1}{I_1} = Z_0 \underline{/\theta} \qquad (233)$$

and

$$\frac{V_2}{I_2} = Z_0 \underline{/\theta} \qquad (234)$$

That is, at both points the voltage and current differ in phase by the same angle . The power at the two points is

$$P_1 = V_1 I_1 \cos\theta \qquad (235)$$

and

$$P_2 = V_2 I_2 \cos\theta \qquad (236)$$

Since $V_2$ and $I_2$ are respectively less than $V_1$ and $I_1$, power has been dissipated in the line in passing from the first point to the second. In any small section of the line, for which the series resistance is r and the leakage conductance g the power dissipated is

$$P = I^2 r + V^2 g, \tag{237}$$

where I is the current through the small section of line and V is the voltage across it. In general, in both power and telephone transmission lines, the leakage conductance is small and the term $V^2 g$ is small compared to $I^2 r$. From this it follows that the power dissipated in the line can be reduced if the current sent over the line is decreased.

Since the power sent over the line is proportional to the product of the voltage andthe current, the current sent over the line can be reduced and the power kept unchanged, if the voltage is correspondingly increased. If the impedance of the line is increased, the ratio of voltage to current is increased. Furthermore, the current required to transmit a given amount of power at a given voltage can be decreased if the phase angle, $\theta$, of the line impedance can be decreased.

In transmitting power at commercial power frequencies, the use of a transformer at the receiving end to step up the impedance of the terminal apparatus increases the impedance of the line and hence increases the ratio of voltage to current on the line, thus reducing the line losses. The transformer, likewise

performs the operation of reducing the voltage to meet the operating requirements of the terminal apparatus. The impedance of power lines is largely determined by the apparatus connected to the line. In general, this apparatus has a positive reactance and in some cases the phase angle of the impedance of the line is reduced by connecting to the line apparatus having a negative reactance, such as synchronous motors, running idle.

A telephone transmission line has, in general, for telephone frequencies such large attenuations that its impedance at the sending end is practically unaffected by the impedance connected across the receiving end. Such a line is said to be electrically "long", while a power line is generally electrically "short" for commercial power frequencies. As a result of the fact that the telephone line is electrically long, use cannot be made of high impedance terminal apparatus to effectively reduce the line losses.

For the usual telephone cable circuit, the expression for the characteristic impedance can be simplified from

$$Z_o = \sqrt{\frac{R + jpL}{G + jpC}} \qquad (196)$$

to

$$Z_o = \sqrt{\frac{R}{jpC}} = \sqrt{\frac{R\angle -90^o}{pC}} = \sqrt{\frac{R}{pC}}\,\angle 45^o$$

because the inductance and conductance are so small as to be negligible. The above expression shows that the impedance of a line varies in magnitude inversely as the square root of the frequency and for all frequencies has a phase angle of -45°. The

current entering the line for a given voltage, therefore, varies with frequency and is out of phase with the voltage by $45^{\circ}$.

If, now inductance can be added to the cable circuit in sufficient amount so that pL is large compared to R, the expression for the impedance can be written

$$Z_o = \sqrt{\frac{jpL}{jpC}} = \sqrt{\frac{L}{C}} \qquad (239)$$

This impedance is indepedent of the frequency and its phase angle is zero, both of which changes are desirable. The magnitude of $Z_o$ will vary as the square root of the inductance and so can be controlled in this way.

The attenuation constant of a cable circuit has been shown to be

$$\alpha = \sqrt{\frac{pRC}{2}} \qquad (213)$$

which varies as the square root of the frequency.

When in a uniform line the series resistance and leakage conductance are small compared respectively with the inductive reactance and the capacity susceptance, an approximate expression for the attenuation constant can be obtained, as follows:

The exact equation for the attenuation constant has been shown to be

$$= \sqrt{\frac{1}{2}\sqrt{\left[(R^2+p^2L^2)\ (G^2+p^2\ C^2) + (GR-p^2LC)\right]}} \qquad (200)$$

Expanding the expression $\sqrt{(R^2+p^2L^2)(G^2+P^2C^2)}$ in this formula by the binominal theorem*, the infinite series is obtai-

*Note: For a discussion of the binominal theorem refer to "A College Algebra" by H.B. Fine, Part II Chapter X.

$$\sqrt{(R^2+p^2L^2)(G^2+p^2C^2)} = p^2LC\left[1+\left(\frac{R}{pL}\right)^2\right]^{\frac{1}{2}}\left[1+\left(\frac{G}{pC}\right)^2\right]^{\frac{1}{2}} = \quad (240)$$

$$= p^2LC\left[1+\frac{1}{2}\left(\frac{R}{pL}\right)^2-\frac{1}{8}\left(\frac{R}{pL}\right)^4 ----\right]\left[1+\frac{1}{2}\left(\frac{G}{pC}\right)^2-\frac{1}{8}\left(\frac{G}{pC}\right)^2 ----\right] =$$

This can be written in the form

$$\sqrt{(R^2+p^2L^2)(G^2+p^2C^2)} = p^2LC + \frac{1}{2}R^2\frac{C}{L} + \frac{1}{2}C^2\frac{L}{C} + \left(\frac{1}{4}\frac{R^2G^2}{p^2LC} ----\right) \quad (241)$$

When, as was assumed, the series resistance and leakage conductance of the line are small compared respectively with the inductive reactance and the capacity susceptance, all terms in equation (241) after the third can be neglected. The expression for the attenuation constant then reduces to

$$\alpha = \sqrt{\frac{1}{2}\left[\frac{1}{2}R^2\frac{G}{L} + GR + \frac{1}{2}G^2\frac{L}{C}\right]} \quad (242)$$

Taking the square root the result is obtained

$$\alpha = \frac{R}{2}\sqrt{\frac{C}{L}} + \frac{G}{2}\sqrt{\frac{L}{C}} \quad \text{See Hill apendix} \quad (243)$$

The above shows, that when inductance is added to a line, and the conductance is assumed to be zero, the attenuation constant assumes the following approximate value

$$\alpha = \frac{R}{2}\sqrt{\frac{C}{L}} \quad (244\text{-A})$$

This can be written

$$\alpha = \sqrt{\frac{RC}{2}\;\frac{R}{2L}} \quad (244\text{-B})$$

Comparing this to equation 213, it is seen that $\alpha$ is reduced by adding inductance, as long as the ratio $\frac{R}{2L}$ is less than $p (p = 2\pi f)$.

As it is not possible to materially increase the inducatance without also adding resistance, the resistance of the circuit is also increased and the beneficial effect of the inductance is partially offset by this increase in the resistance. It is, however, practicable to get a material net benefit from this addition of inductance of "loading" as this addition has been designated. If the conductance is not zero, the attenuation constant is by equation (243)

$$\alpha = \frac{R}{2}\sqrt{\frac{C}{L}} + \frac{G}{2}\sqrt{\frac{L}{C}} \qquad (243)$$

This can be written also

$$\alpha = \sqrt{CL}\left(\frac{R}{2L} + \frac{G}{2C}\right) \qquad (243\text{-a})$$

In the last formula $\frac{R}{2L}$ is called the "damping constant" of the series constants of the line and $\frac{G}{2C}$ the "damping constant" of the shunt constants of the line. It is seen that the attenuation, and, therefore, the line losses are proportional to the sum of these two damping constants.

The corresponding value for the wave length constant is

$$\beta = pn\sqrt{CL} \qquad (245)$$

and the velocity is

$$W = \frac{1}{\sqrt{CL}} \qquad (246)$$

The attenuation as expressed by formula (243) is independent of frequency. Actually, however, the attenuation of a loaded line varies with frequency largely because the resistance introduced by the added inductance, increases with

frequency and also the conductance increases with frequency .

It is practicable to add inductance uniformly to a line. This is generally done by winding around the conductors a layer of iron wire or tape. The amount of inductance which can be added practically in this manner is, however, limited. For many purposes, a much better method of increasing the inductance of the circuit is the periodic insertion of inductance coils in the circuit. Such an addition of inductance is referred to as "lumped loading" * to distinguish it from the uniform loading described above.

*Note: for treatment of theory of loading --Refer to "Propagation of Electric Currents in Telephone and Telegraph Conductors", J.A. Fleming, Chapter IV; Philosophical Magazine, Vol. V, Page 319, March, 1903; G. A. Campbell, Trans. A.I.EE., Vol 16, Page 93, 1899 and Vol. XVII, Page 445, 1900, M.L. Pupin.

2. Lumped Loading.

Figure 43 illustrates the application of lumped loading to a cable circuit in which the length of cable between two adjacent loading coils is represented by a network consisting of the lumped resistance and capacity of the section of cable. In the case of telephone cables such a network is a fair approximation for sections of cable up to a mile or two in length. Figure 44 represents one "loading section" of the loaded circuit, which loading section is terminated at each end in a "half loading coil",

Substituting in equations (92) and (93) the corresponding values for the circuit of figure 44, the ratio of the currents at the two ends of the section is

$$\frac{I_2}{I_1} = \frac{-j\frac{1}{pC}}{\left(\frac{R}{2} + j\frac{pL}{2}\right) - j\frac{1}{pC} + \sqrt{-j\frac{1}{pC}(R + jpL) + \frac{(R+jpL)^2}{4}}} \qquad (247)$$

To simplify the consideration of this equation it will be assumed that R can be eliminated. This then reduces to

$$\frac{I_2}{I_1} = \frac{-j\frac{1}{pC}}{j\frac{pL}{2} - j\frac{1}{pC} + \sqrt{\frac{L}{C} - \frac{p^2 L^2}{4}}} \qquad (248\text{-a})$$

Dividing by $-j\frac{1}{pC}$

$$\frac{I_2}{I_1} = \frac{1}{1 - \frac{p^2LC}{2} + j\,p\sqrt{LC}\sqrt{1 - \frac{p^2LC}{4}}} \qquad (248\text{-b})$$

Equation (170) showed that the resonance frequency $f_r$ for a combination of L and C is

$$f_r = \frac{1}{2\pi\sqrt{LC}} \tag{170}$$

or,

$$\sqrt{LC} = \frac{1}{2\pi f_r} \tag{249}$$

Substituting this value for $\sqrt{LC}$ in equation (248-b) and remembering that $p = 2\pi f$, the expression is obtained:-

$$\frac{I_2}{I_1} = \frac{1}{1 - \frac{1}{2}\left(\frac{f}{f_r}\right)^2 + j\left(\frac{f}{f_r}\right)\sqrt{1-\left(\frac{f}{2f_r}\right)^2}} \tag{250}$$

As long as f is less than $2f_r$ the term under the square root sign is positive. If f is greater than $2f_r$ this term becomes negative and can be written:-

$$\sqrt{-1\left[\left(\frac{f}{2f_r}\right)^2 - 1\right]} = j\sqrt{\left(\frac{f}{2f_r}\right)^2 - 1} \tag{251}$$

Equation (250) then becomes for f greater than $2f_r$

$$\frac{I_2}{I_1} = \frac{1}{1 - \frac{1}{2}\left(\frac{f}{f_r}\right)^2 - \left(\frac{f}{f_r}\right)\sqrt{\left(\frac{f}{2f_r}\right)^2 - 1}} \tag{252}$$

Referring to equation (250) the magnitude of the denominator is:-

$$\left[1 - \frac{1}{2}\left(\frac{f}{f_r}\right)^2\right]^2 + \left[\frac{f}{f_r}\sqrt{1 - \left(\frac{f}{2f_r}\right)^2}\right]^2 =$$

$$= 1 - \left(\frac{f}{f_r}\right)^2 + \frac{1}{4}\left(\frac{f}{f_r}\right)^4 + \left(\frac{f}{f_r}\right)^2 - \frac{1}{4}\left(\frac{f}{f_r}\right)^4 + \ldots \tag{253-a}$$

$$= 1 \tag{253-b}$$

For frequencies less than $2f_r$, $I_2$ equals $I_1$ in magnitude but differs in phase.

For $f = 2f_r$, the ratio becomes

$$\frac{I_2}{I_1} = \frac{1}{1 - 2} = -1 \qquad (254)$$

and $I_2$ is therefore equal to $I_1$ but is opposite in direction.

For grea..er

For f greater than $2f_r$, the denominator of equation (252) becomes greater than the numerator, or $I_2$ is less than $I_1$.

$$\text{For } f = 3f_r \quad I_2 = \frac{I_1}{6.86} = .146\, I_1$$

$$\text{"} \quad f = 4f_r \quad I_2 = \frac{I_1}{13.9} = .072\, I_1$$

$$\text{"} \quad f = 5f_r \quad I_2 = \frac{I_1}{23.0} = .044\, I_1$$

From these expressions it is seen that for all frequencies up to a frequency of $f = 2f_r$ the attenuation is zero, while for frequencies beyond that value the attenuation is very large. By taking $f = 2f_r$, this frequency, which is called the critical frequency of the loading, $f_c$, is obtained:

$$f_c = \frac{1}{\pi \sqrt{LC}} \qquad (255)$$

Where L is the inductance of a loading coil and C is the total capacity of the length of cable between adjacent coils, i.e., the capacity of a loading section. If the coils are spaced S miles apart and $C_m$ is the capacity per mile, equation (255) becomes:-

$$f_c = \frac{1}{\pi \sqrt{L\, S\, C_m}} \qquad (256)$$

It is seen that a system of periodic lumped loading will transmit all frequencies up to the value given by equation (252). For frequencies below this value the lumped loaded system can be considered, in the ideal case, the equivalent to the uniformly loaded system of the same inductance per mile. This relation is modified, however, by the resistance of the circuit and loading coils, which was eliminated in considering equation (244). This resistance causes attenuation for frequencies below the critical frequency and also causes the attenuation to increase as the critical frequency is approached. For frequencies up to about 80 per cent. of the critical frequency, the approximation of the lumped loaded to uniformly loaded lines is very close. For the lumped loaded line the range of frequencies below the critical frequency is called the free range and the line is said to "cut off" at the critical frequency; that is, its transmission stops at that frequency.

The velocity of propagation of the current along the line expressed in terms of the number of loading sections passed per second is given by the following (refer to equation 246).

$$W = \frac{1}{\sqrt{CL}} = \frac{1}{\sqrt{S\ C_m\ L}} \qquad (257)$$

Where S is the distance between loading coils in miles, $C_m$ the capacity per mile of the line and L the inductance of a loading coil.

The critical frequency can then be expressed

$$f_c = \frac{W}{\pi} \qquad (258\text{-a})$$

or, $$\frac{W}{f_c} = \pi \qquad (258\text{-b})$$

Since $\frac{W}{f_c}$ is the wave length at the critical frequency, this wave length is $\pi$ loading section long, or there are $\pi$ loading coils per wave length. For all frequencies below $f_c$ there are more than $\pi$ coils per wave length. This is another way of stating the condition for lumped loading being equivalent to uniform loading.

Referring to equation (256)

$$f_c = \frac{1}{\pi \sqrt{L\ S\ C_m}} \qquad (256)$$

It is seen that $f_c$ can be controlled by varying the product $L\ S\ C_m$, or since in any circuit $C_m$ is fixed by varying L, the inductance of the loading coils, or S, their spacing in the circuit. In laying out loading for telephone circuits, it has been assumed in many cases that a critical frequency of 2,200 cycles gives satisfactory transmission. If this value is used, the product $L\ S\ C_m$ is fixed.

$$W = \frac{1}{\sqrt{L\ S\ C_m}} = \pi f_c = 7000 \text{ approximately} \qquad (259)$$

For this critical frequency, the circuit is said to be loaded to a velocity of 7000 "loads" per second. The product of $L\ S\ C_m$ is fixed but the ratio of $\frac{L}{S\ C_m}$ can be varied. Referring to equation (239) this shows that the characteristic impedance $Z_0 = \sqrt{\frac{L}{C}}$ can be controlled, and hence the ratio of line voltage to current and so the losses in the line. For large values of $Z_0$, large values of coil inductance are used with a short spacing. For lower values of $Z_0$ the inductance of the loading coils is decreased and the spacing increased.

From equation (259) the spacing to be used for any given inductance of loading to meet a given critical frequency is:-

$$S = \frac{1}{W^2 L C_m} \quad \text{or} \quad \frac{1}{\pi^2 f_c^2 L C_m} \qquad (260)$$

Thus the inductance is fixed by the desired impedance of the line and S is then determined by the above equation.

Since the loading coils introduce resistance into the circuit, their loading effect is offset to some extent by the losses in the resistance of the coils. For any given loading coil structure this resistance increases with the inductance. This factor, together with the cost of the coils, makes it usually uneconomical to add the amount of inductance required to attain the condition for the ideal distortionless line when RC = LG. (Refer to equation 209-b).

3. Filters.

The property of the lumped loaded line of transmitting all frequencies below a predetermined frequency and of presenting large attenuation to all frequencies above this critical frequency is utilized in networks of inductance and capacity in cases where such discrimination between ranges of frequencies is desired. Networks which can be so designed as to discriminate between frequency ranges are called "filters".* Such a periodic network, which consists of series inductances and shunt capacities is shown in figure 45. It corresponds closely to the lumped loaded line discussed above, and the treatment of the loaded line on the basis that the line constants are lumped applies directly to the network of figure 45. In accordance with the formulae deduced for the lumped loaded line, this network can be designed to meet any given critical frequency and impedance. The filter shown in figure 45 is called a "low pass filter" because it transmits the lower frequencies and eliminates the higher frequencies.

Figure 46 shows a periodic network in which the series elements are capacities and the shunt elements inductances. Such a filter will present a high attenuation for all frequencies below the critical frequency and will transmit all frequencies above this frequency. Such a filter is called a "high pass filter". Its critical frequency is given by the following formula:

$$f_c = \frac{1}{4\sqrt{LC}} \qquad (261)$$

* NOTE: For a discussion of wave filters refer to:
U.S. Patent No. 1, 227, 113 (May 22, 1917)
G. A. Campbell.

Other more complex networks can be designed to transmit only a band of frequencies, eliminating all frequencies outside of the band, and are known as "band filters".

## III. TELEPHONE TRANSMISSION

The problem of telephone transmission is to transmit speech sounds over the telephone system so that the sounds delivered at the receiving end will be sufficient in volume to be easily heard and will bear sufficiently close resemblance to the sounds impressed at the transmitting end to be readily recognized.

Practical telephone systems, because their efficiency is in general materially below 100 per cent., do not deliver from the receiver as great a volume of sound as was impressed upon the transmitter. Also, the efficiency of the system varies with frequency and so distorts the speech sounds. These two factors, volume and distortion, together with the extraneous interference, such as noise on the line and noise in the room where the telephone set is located, affect the degree of satisfaction obtained in the use of the telephone. This degree of satisfaction is referred to as the "quality" of the telephone system. As a certain grade of transmission has been found to give satisfactory commercial service it is part of the transmission problem to establish means for providing and maintaining such service at the minimum cost. This grade of service varies somewhat with the conditions and is affected to a large extent by the cost of giving different grades of service.

### 1. Speech sounds.

Speech sounds are vibrations in the air set up in the throat, mouth and nose cavaties in talking. These sounds are

extremely complex. For undistorted transmission they require a circuit capable of transmitting with equal efficiency all frequencies from about 100 to 8,000 cycles. Practically, however, easily intelligible speech can be obtained with a circuit transmitting a band of frequencies between 200 and 2,000 cycles. The extension of this band to higher frequencies increases, of course, the ease of understanding the sounds. At the present time, however, economical and practical considerations do not justify the use of a much wider band than 2,000 cycles. Improvements in the system, however, may make desirable an increase in this range.

Speech sounds may be roughly grouped into two classes, vowel sounds and consonant sounds. The fundamental sounds of the English language, so grouped, are listed in the following table. In general, vowel sounds can be sustained while consonants are more transient in their nature, being used as methods of starting and stopping vowel sounds.

Classification of the Fundamental Sounds of English Speech

| Vowel Sounds | Key Word | Consonant Sounds | Key Word |
|---|---|---|---|
| a | top | b | ball |
| ā | tape | ch | cheap |
| a̍ | tap | d | day |
| e | ton | f | fall |

## Classification of the Fundamental Sounds of English Speech (Continued)

| Vowel Sounds | Key Word | Consonant Sounds | Key Word |
|---|---|---|---|
| ē | team | g | gold |
| e̍r | term | h | hence |
| i | tip | j | jump |
| ī | time | k | keep |
| o | ton | l | look |
| ō | tone | m | man |
| o̍ | talk | n | no |
| u | took | ng | sing |
| ū | tool | p | pay |
| ou | town | r | red |
| | | s | say |
| | | sh | ship, azure |
| | | th | then, thin |
| | | t | ten |
| | | v | view |
| | | w | we, why |
| | | y | you |
| | | z | zest |

Totals 14 vowels

22 Consonants

36

2. Telephone Circuit

As is known, the telephone circuit operates by converting the sound waves into electrical waves, transmitting the electrical waves and then converting these electrical waves into sound waves. As these waves are all complex in their nature, it is practically impossible to use them directly in dealing with transmission problems. It has been found that most telephone transmission problems may be handled on the basis that the speech waves are equivalent to a combination of a number of alternating currents of different frequencies. This makes the telephone transmission problem one of transmitting a band of frequencies. The efficiency of a telephone circuit is then investigated by determining the efficiency with which different frequencies in this band are transmitted over the circuit, and many problems can be satisfactorily attacked on the basis of using a single frequency as representative of this range. For purposes of comparing the volume of speech sounds received over two circuits, it has been found that in many cases the transmission over the circuits of current of about 800 cycles is indicative.

All telephone circuits involve at least three elements, the transmitter, the receiver, and the line connecting the two. In practice at least two other elements are generally involved, a substation circuit to provide efficient two-way transmission, and a circuit for providing direct current to energize the transmitter.

3. Transmitter.

The transmitter is a means of converting sound energy into electrical energy. The usual device employed for this purpose

consists of a diaphragm on which the sound waves impinge and which, by its vibrations, changes the resistance of carbon contacts by varying the pressure on these contacts. These contacts being in a direct current circuit, change the resistance of this circuit and cause variations in the direct current. The variations in the direct current are thus, in the ideal case, proportional to the variations in the sound waves.

The direct current can be looked upon as a means of energizing the transmitter. Its function, consequently, is in a way similar to that of the field current of a generator. In a transmitter the measure of efficiency is the ratio of the varying electrical power output to the power of the sound waves striking the diaphragm. In general, in commercial carbon button transmitters, the electrical output is greater than the sound power input, that is, the transmitter is an amplifying device. In studying telephone circuits, it is convenient to consider the transmitter as a generator of varying currents with an internal resistance.

4. Receiver.

The receiver is a device for converting electrical energy into sound energy. The receiver usually employed consists of a winding of wire around a magnet arranged so that, as the current through this winding varies, corresponding changes are produced in the field of the magnet. This field changes the attraction which the magnet exerts on a diaphragm, causing the latter thus to vibrate in accordance with the current variations and to produce sound vibrations in the air. This type of receiver can be compared to the a.c. induction motor in which the diaphragm corresponds to the rotor.

5. Substation Set.

In addition to the fact that the telephone transmission circuit differs from the power circuit in that it is required to transmit a band of frequencies instead of only a single frequency, the telephone circuit is a two-way circuit. It must be possible to talk and listen at the same point. It would, of course, be possible to set up two complete one-way circuits to operate in opposite directions between two points and so provide two-way transmission. This practice has been adopted for some of the toll circuits, but in view of what can be accomplished with a two-way circuit, it is uneconomical, in general, to employ two one-way circuits throughout the telephone plant for the circuits between the subscriber and the central office and for the local trunks between central offices not widely separated.

The substation set provides a means of carrying on a two-way conversation over two wires without any manipulation on the part of the telephone user to switch the circuit for transmitting at one time and receiving at another. A substation circuit which permits this two-way operation is called an "invariable" set.

The simplest form of a substation set is merely a transmitter and a receiver connected in series and the combination connected to the line. This arrangement is shown in Figure 47. Some arrangement needs to be provided for supplying direct current to the transmitter. This can be done by placing a battery in serie with the line and set.

Assuming that the impedances of these three elements, the transmitter, receiver and line are all pure resistances at telephone frequencies, consideration will be given to the requirements of the circuit and their effect on the design. For maximum power transfer between the line and the set, the two should be equal, that is

$$L = T + R \qquad (262)$$

In transmitting, the transmitter can be considered as a generator of internal resistance "T" which sends current through the receiver and the line in series. The power delivered to the receiver in this case is wasted in so far as transmission to the far end of the line is concerned. Also, on receiving the transmitter is in series with the receiver and the power delivered to the transmitter is wasted.

The transmitting efficiency of any substation set is defined as the ratio of the power actually sent into the line from the transmitter in the set to the maximum power that could be sent into a line by the transmitter.

Similarly, the receiving efficiency of a substation set is defined as the ratio of the power actually sent into the receiver of the set from the line to the maximum power that could be sent into a receiver by the line.

The condition for maximum power transfer, as has been shown, requires that the generating element be equal to the element to which power is to be delivered. This requires that on transmitting the line be equal to the transmitter, and on receiving the receiver be equal to the line. For the simple series set of Figure 47, this

would mean that the receiver would need to have zero impedance when the set is used for transmitting and the transmitter have zero impedance when the set is receiving.

The transmitting efficiency then follows:

$$F_T = \frac{4\,L\,T}{(T+R+L)^2} \qquad (263)$$

and for the receiving efficiency the expression is obtained

$$F_r = \frac{4\,L\,R}{(T+R+L)^2} \qquad (264)$$

Since by equation (262) $R+T = L$, it follows that

$$F_T = \frac{T}{L} \qquad (265)$$

and

$$F_R = \frac{R}{L} \qquad (266)$$

The combined efficiency, being the product of the two, is:

$$F = \frac{T\,R}{L^2} \qquad (267)$$

In Figure 48 the total efficiency F is plotted against the ratio $\frac{F_T}{F_R} = \frac{T}{R}$. From this plot it will be seen that maximum total efficiency is obtained when

$$T = R \qquad (268)$$

For this condition, it follows that

$$F_T = F_R = \frac{1}{2} \qquad (269)$$

$$F = F_T\,F_R = \frac{1}{4} \qquad (270)$$

Of the total power generated on transmitting, one-half is delivered to the line. In receiving, of the power delivered to the set, one-half gets to the receiver and one-half is wasted in the transmitter. Thus the power efficiency of the set in transmitting is 50 per cent. and likewise 50 per cent. on receiving. The combined transmitting and receiving efficiency of the set is, therefore, 25 per cent. which is the maximum possible to obtain with a two-way set.

Practically all receivers and most lines have reactance as well as resistance, so that it is not possible, unless the reactances can be neutralized to obtain the maximum (25 per cent) efficiency of speech transmission. This result might possibly be accomplished at one frequency, but there is no practical way of neutralizing the reactance at all telephone frequencies, so that actually the efficiency of a set is much below the maximum that is theoretically possible.

It has not been possible to construct a commercial transmitter of suitable efficiency which has resistance as great as one-half of the line impedance. For this reason a transformer or induction coil is generally used in telephone sets to step up the impedance of the transmitter so as to approximately fulfill the impedance requirements given above.

The simplest case of such use is shown in figure 49 which gives the circuit for the standard local battery substation set. Here the transformer is used to step up the impedance of the transmitter. The impedance of the receiver is approximately half of the line impedance without a transformer.

The induction coil in the standard common battery substation set, shown in figure 50, is more complex in its effect but the circuit is designed so that the inpedance of the transmitter and receiver are adjusted to the line. The condenser in the circuit, in addition to preventing the direct current supply for the transmitter from flowing through the receiver, also compensates for some of the inductive reactance of the receiver.

It will be noted that on transmitting a fourth of the total energy generated is delivered to the receiver of the transmitting set, and causes the receiver to operate. This operation of the receiver in the transmitting set by the current generated by the transmitter is called "side tone".

This side tone effect is at times objectionable in that noises in the room excite the transmitter and produce sounds in the receiver. Furthermore, the speaker talks in his own ear, and if he talks very loud this may be uncomfortable to his ear. As the efficiency of the transmitter and receiver is increased, these effects become more serious. It is possible to reduce and in some cases eliminate this effect by the use of what is called an "anti-side tone set." Such a set is standard for operators' use. The standard operator's set circuit is shown in figure 55.

This condition of having a two-way circuit in which the transmitting element does not affect the receiving element is important in any relay or repeater in a telephone or telegraph circuit. The principle of the anti-side tone telephone set is the same as that of the telephone repeater circuit and of the telegraph duplex set. All of them are analogous to the Wheatstone bridge circuit which has been discussed previously.

Refer now to figure 51 which is the same as figure 36, treated previously. It is known that if

$$\frac{R_3}{R'_2} = \frac{R_2}{R'_3} \qquad (86\text{-a})$$

no current flows in $R_1''$ for a voltage in series with $R_1$.

This relation also holds for alternating current and impedances, so circuit 51 can be changed to circuit 52, in which the resistances have been replaced by impedances. The relation

$$\frac{Z_3}{Z_2'} = \frac{Z_2}{Z_3'} \qquad (271)$$

is obtained which can also be written

$$Z_3 \; Z_3' = Z_2 \; Z_2' \qquad (272)$$

For simplification make $Z_2' = Z_3'$ and $Z_2 = Z_3$.

In figure 53 a transmitter T has been substituted for $Z_1$. a receiver R for $Z_1'$ , a line L for $Z_3$, and impddance N for $Z_2$ and inductive reactances $X_1$ and $X_2$ for $Z_2'$ and $Z_3'$.

Then if,

$$\frac{X_1}{X_2} = \frac{L}{N} \qquad (273)$$

no current will flow in the receiver R for a voltage in the transmitter T.

Figure 54 shows the two inductances replaced by two equal transformers and figure 55 combines these two transformers into one with the middle point brought out of the secondary winding. The latter arrangement represents the standard operator's set circuit. This is practically the same circuit as is used in telephone repeaters.

The introduction of the fourth element N which is equal in impedance to the line, makes possible an arrangement in which the anti-side tone condition is obtained.* Furthermore, this circuit is inherently just as efficient as the side tone circuits, that i .

*NOTE: For a comprehensive discussion of substation circuits refer to a paper presented before A.I.E.E., February 19, 1920, entitled, "Maximum Output Networks for Telephone Substation and Repeater Circuits," by G.A. Campbell and R.M. Foster.

50 per cent. in transmitting and 50 per cent. in receiving. In transmitting no power is furnished to the receiver but is delivered to L and N. In receiving from the line no power is furnished to N, but is delivered to T and R. In the arrangement shown, for maximum efficiency the impedance of R should still equal one-half L. Since the number of turns on the primary winding can be changed without affecting the balance of the circuit, this primary winding can be adjusted to meet any resistance transmitter.

As was pointed out in connection with the Wheatstone bridge T and R can be interchanged without affecting the balance.

<u>6. Standard Reference Circuit.</u>

The simple telephone circuit shown on drawing No. 38-Y-724 has been adopted as a standard to which other telephone circuits can be compared in order to determine their efficiency. The circuit consists of two standard common battery substation sets connected through two 25-A repeating coils to an adjustable length of artificial cable. The substation sets use a No. 229 standard transmitter, a No. 122 standard receiver, a No. 20 standard induction coil and a 2-mf. condenser. Twenty-four volt battery is supplied to the transmitters through the No. 25-A repeating coils to which the sets are connected. The artificial line represents a standard No. 19 gauge cable having 88 ohms resistance and .054-mf. capacity per loop mile. The attenuation constant of this cable at 800 cycles is $\alpha = .109$.

In determining the efficiency of a telephone circuit in terms of this standard circuit, both circuits are talked over in

turn and the amount of artificial cable in the standard circuit is adjusted to give equal received volumes of sound over the two circuits. The amount of cable inserted in the standard circuit to give this balance is then stated as the "transmission equivalent" of the circuit tested. The equivalent of this circuit is thus expressed as a certain number of "miles of standard cable".

The effect of any change in the circuit tested, such as the insertion of a piece of cable or a relay, or the substitution of another transmitter or receiver, can be determined by the change in the length of artificial cable in the standard circuit required to obtain a volume balance. If this change has resulted in decreasing the volume of sound received over the circuit under test, the amount of artificial cable in the standard circuit will need to be increased for the new balance, say by 1.5 miles. The change is then said to cause a "transmission loss" of 1.5 miles of standard cable. Likewise, a decrease in the length of artificial cable of 1.5 miles would signigy a transmission gain of 1.5 miles. In this way the transmission efficiency of any piece of apparatus or line in a telephone circuit can be expressed in terms of the length of standard cable which, when inserted in a trunk of standard cable, causes the same change in the received power.

From the consideration of the insertion of impedance elements into networks, it is evident that the magnitude of the "transmission loss" caused by inserting any piece of apparatus of line into a telephone circuit is dependent upon the impedance characteristics ofthe circuit in which it is placed and so the

loss will vary with the circuit. It is, therefore, in general, necessary to specify the conditions for which the loss due to a particular piece of apparatus applies. In some cases, the possible range of variation which can be obtained under practical conditions is small and a fixed value can be chosen for the loss which can be used as a close approximation for all condition met with in practice.

The basis for expressing the transmission efficiency of a piece of telephone equipment, as "n" miles of standard cable, is that the piece of equipment causes a reduction in the power received over the circuit in which it is placed, which is the same as the difference in the amounts of power at two points along a length of standard cable "n" miles apart. The ratio of the amounts of power at these two points is equal to the square of the ratio of the currents at these two points, and it has been shown that this current ratio can be expressed in terms of the attenuation per mile of the circuit, that is

$$\frac{I_2}{I_1} = e^{-n\alpha} \qquad (274)$$

The square root of the ratio of the amount of power received over the telephone circuit with the piece of equipment in the circuit to the amount of power received when it is not present, has been called the "equivalent current ratio" and used as a measure of the change produced in the circuit. From this equivalent current ratio, the corresponding attenuation can be obtained. This attenuation divided by the attenuation constant of standard cable for a

particular frequency gives the equivalent miles of standard cable at this frequency. Where the transmission problem is dealt with on a single frequency basis, it is customary to use the attenuation constant of standard cable for 800 cycles, and the equivalent is thus obtained in terms of "miles of standard cable at 800 cycles," or "800-cycle miles."

To simplify the work of obtaining this equivalent in terms of miles, the attached drawing No. 701-8740 has been prepared which gives directly the number of "800-cycle miles" which corresponds to any given current ration.

7. Distinction between Telephone and Power Transmission.

As has been indicated, the telephone transmission problem differs from the power transmission problem in that the former involves the transmission of a band of frequencies and, in general, a two-way circuit must be provided. One other fundamental distinction between the two is brought out by the discussion of efficiency and maximum power transfer.

The function of the power system is to supply a given amount of power at a given point at a minimum expense. With the low frequencies involved, it is practicable to make the line losses low. Since the main cost of delivering the given amount of electical power is the cost of generating power at the power house, it is necessary for economy to keep the ratio of delivered to generated power high. The power plant is therefore designed so as to keep down the cost of running the prime movers which are turning over the generators. Also, in the power system, the endeavor is to

supply a fairly constant voltage at all points along the system and, therefore, the voltage drop in the generator should be kept low so as to maintain the voltage at the load terminals independent of the load.

In the telephone system the power of the prime mover is the power in the speech sounds. This is under the control of the user and not of the owner of the system. It is necessary to furnish the telephone subscriber with a means for converting this speech power into electrical power, that is, thetransmitter. It is, of course, important that the transmitter carry this function out as efficiently as possible. The problem then is to get as large an amount of this electrical power from the transmitter to the line, and, after having transmitted it over the line, to get the maximum amount into the receiver at the far end. The line losses are, in general, necessarily large and the impedance of the receiver has practically no effect on the power put into the line by the transmitter. Thus, the matter of obtaining the condition for maximum power transfer from one element to another in the circuit is of great importance. In the power system the problem is to keep down the cost of supplying a given amount of power. In the telephone system, driving power is furnished by the talking subscriber and the problem is to deliver the maximum amount to subscriber listening at the far end of the circuit.

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# FIGURES TO ACCOMPANY NOTES ON FUNDAMENTALS OF TRANSMISSION




FIG 1


FIG. 2

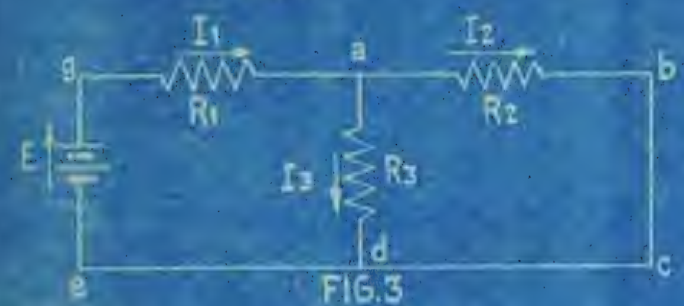


FIG. 3

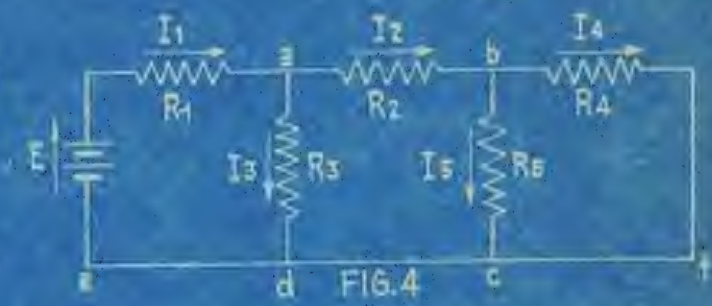


FIG. 4

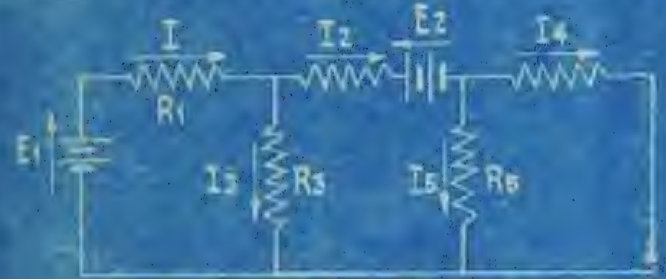


FIG. 5


FIG. 6

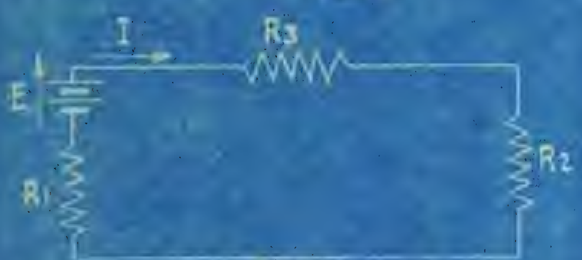


FIG. 7


FIG. 8

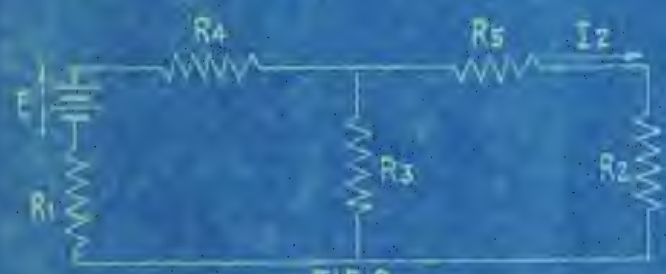


FIG. 9


FIG. 10


FIG. 11


FIG. 12

E.D.
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# FIGURES TO ACCOMPANY NOTES ON FUNDAMENTALS OF TRANSMISSION



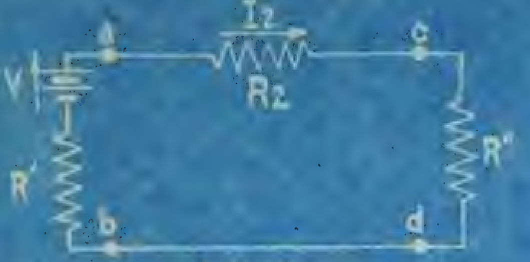

FIG.13

FIG.14

FIG.15

FIG.16

FIG.17

FIG.18

FIG.19

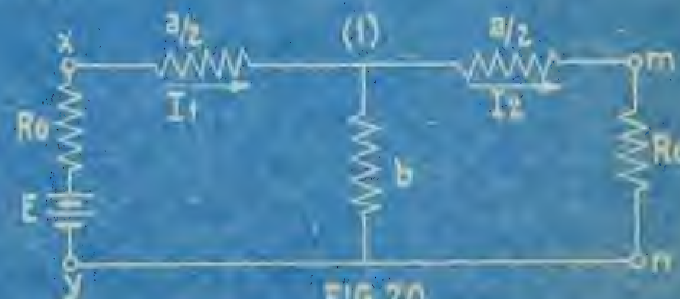

FIG.20

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# FIGURES TO ACCOMPANY NOTES ON FUNDAMENTALS OF TRANSMISSION


FIG. 21

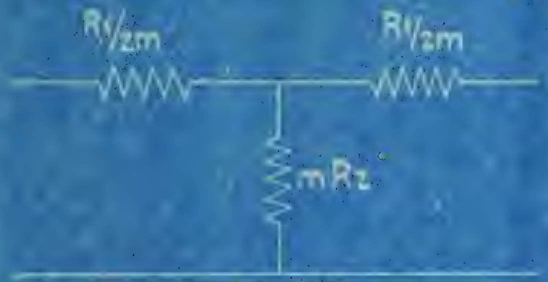


FIG. 22

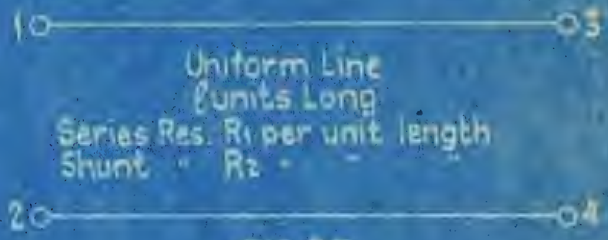


FIG. 23

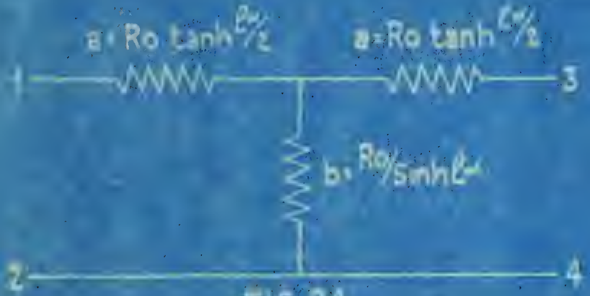


FIG. 24

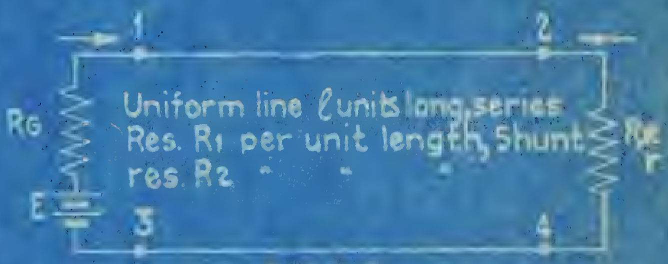


FIG. 25

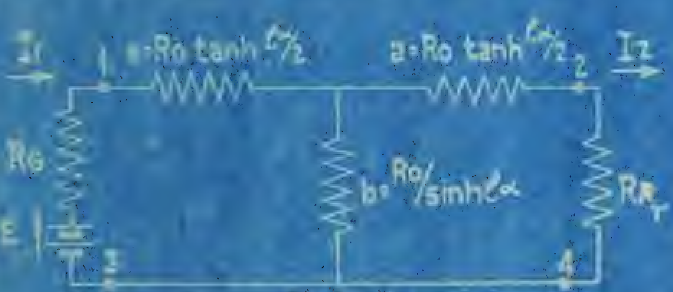


FIG. 26

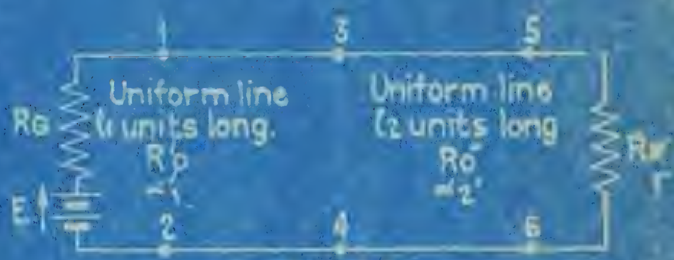


FIG. 27

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# FIGURES TO ACCOMPANY NOTES ON FUNDAMENTALS OF TRANSMISSION


Fig. 28

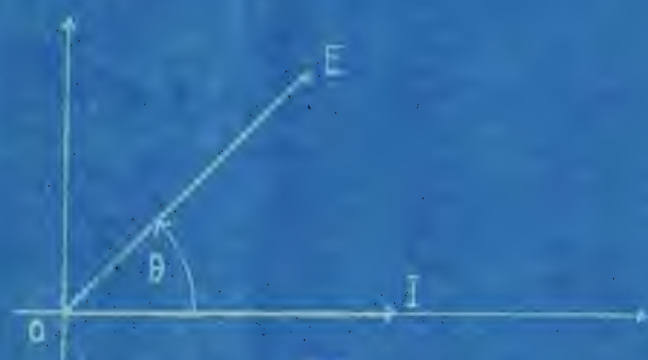


Fig. 29

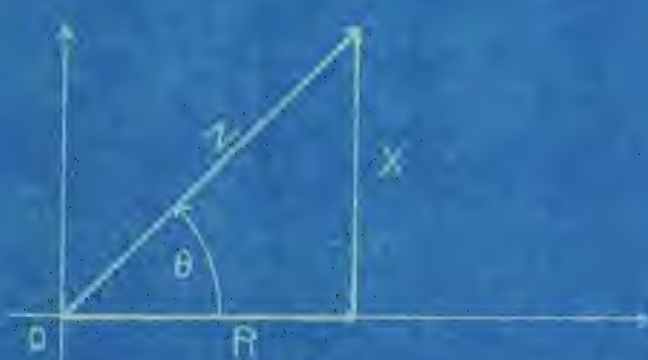


Fig. 30

R.D.
10-5-20

# FIGURES TO ACCOMPANY NOTES ON FUNDAMENTALS OF TRANSMISSION



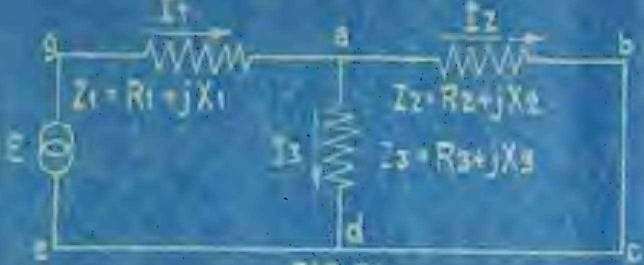


FIG. 31

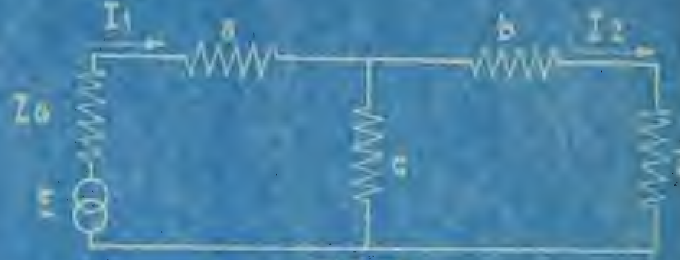


FIG. 35


FIG. 32

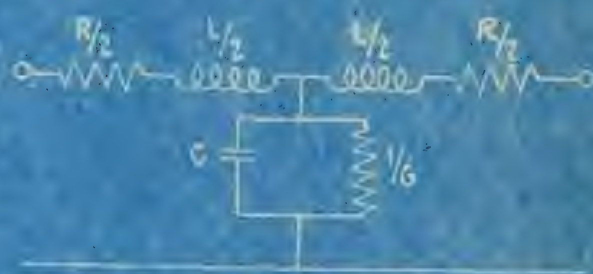


FIG. 36

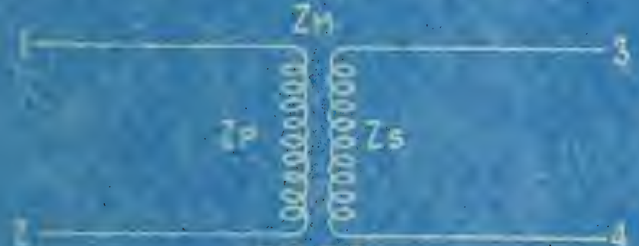


FIG. 33


FIG. 42

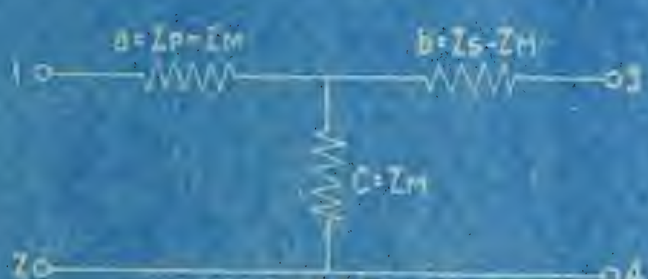


FIG. 34

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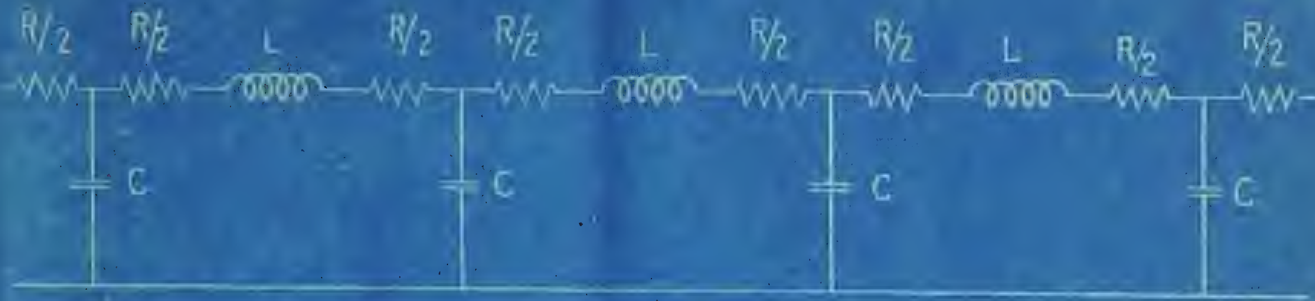


Fig. 43

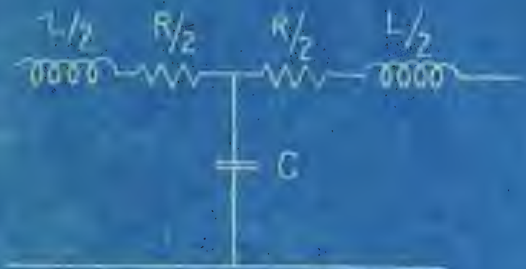


Fig. 44

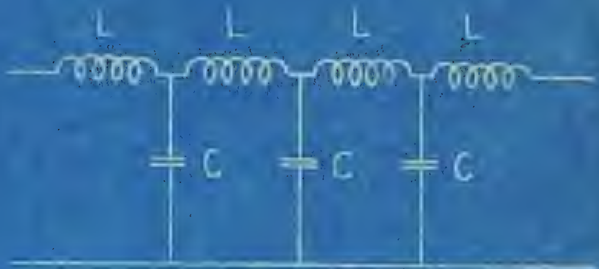


Fig. 45

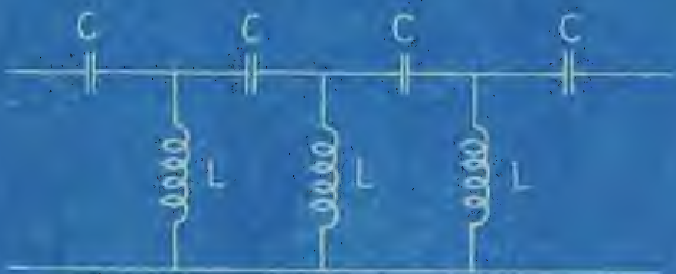


Fig. 46

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Fig. 47


Fig. 49


Fig. 50

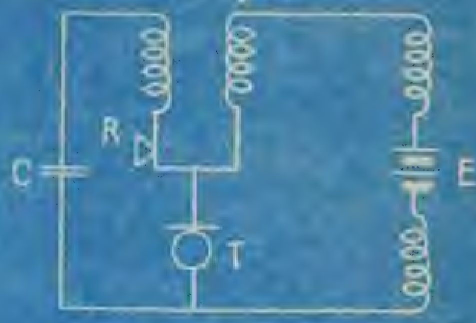



Fig. 51


Fig. 52

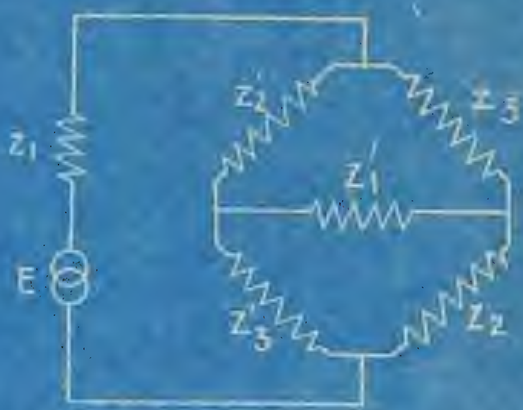



Fig. 53

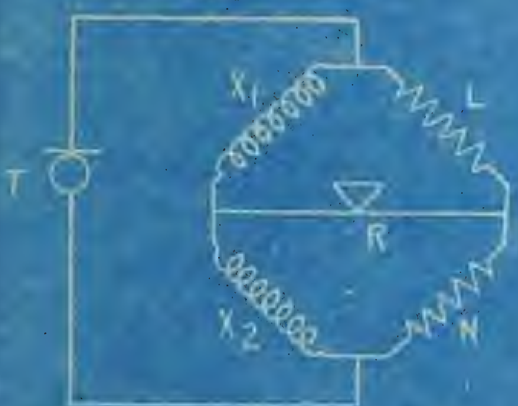



Fig. 54

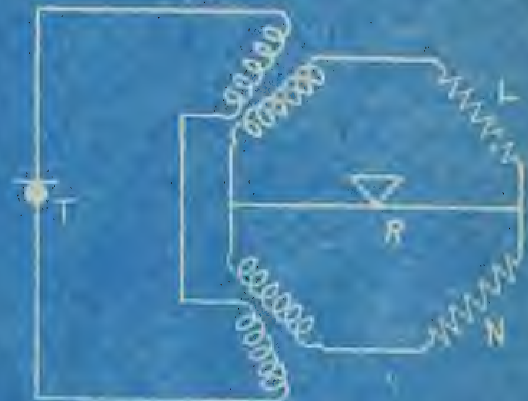
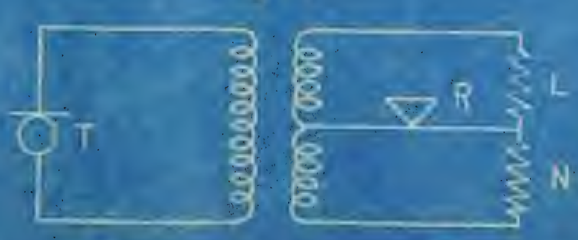


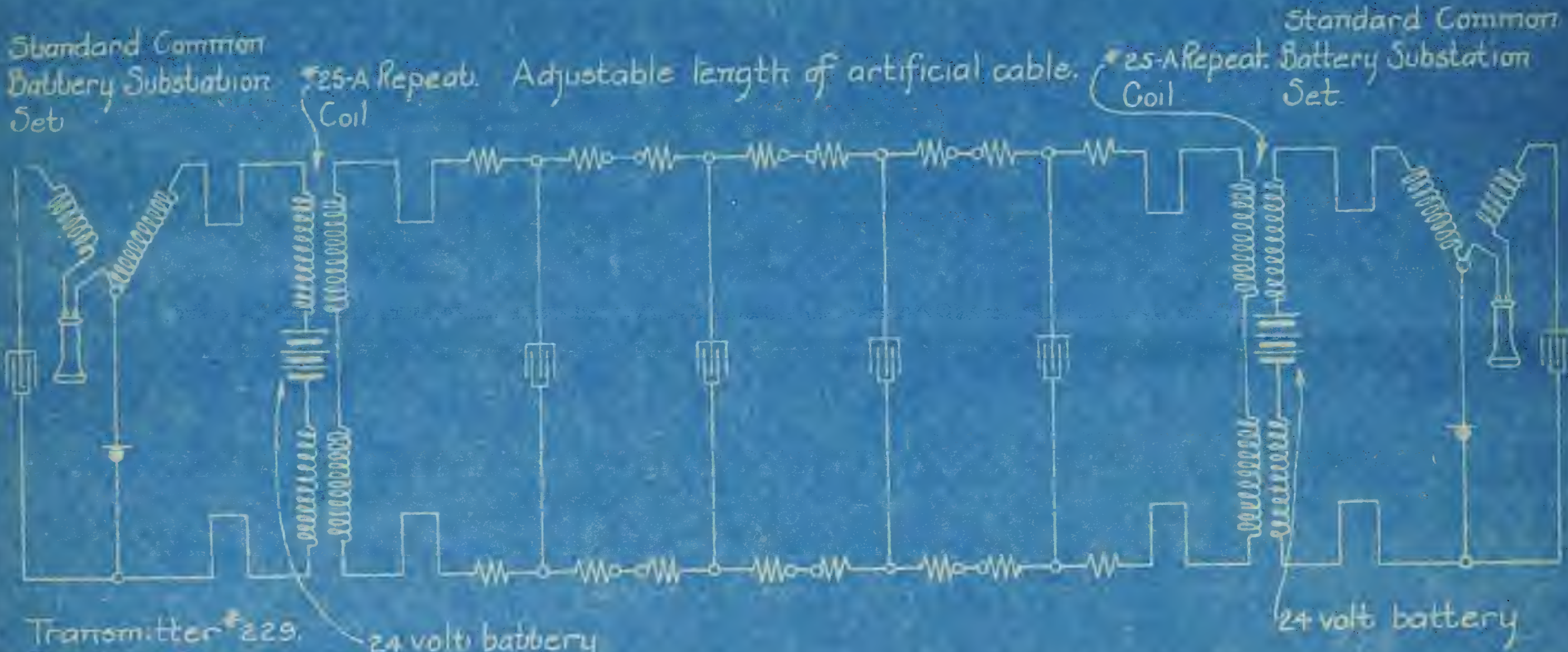

Fig. 55




10-7-20

# STANDARD CIRCUIT.

For Transmission Work.




Transmitter #229.
Receiver # 122.
Induction Coil #20
Condenser 2 mf'd.

38-Y-724
Jan. 29, 1912.

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FIGURES TO ACCOMPANY NOTES ON FUNDAMENTALS OF TRANSMISSION

CURVES SHOWING INSTANTANEOUS CURRENT
ALONG NO. 8 GAUGE OPEN WIRE LINE

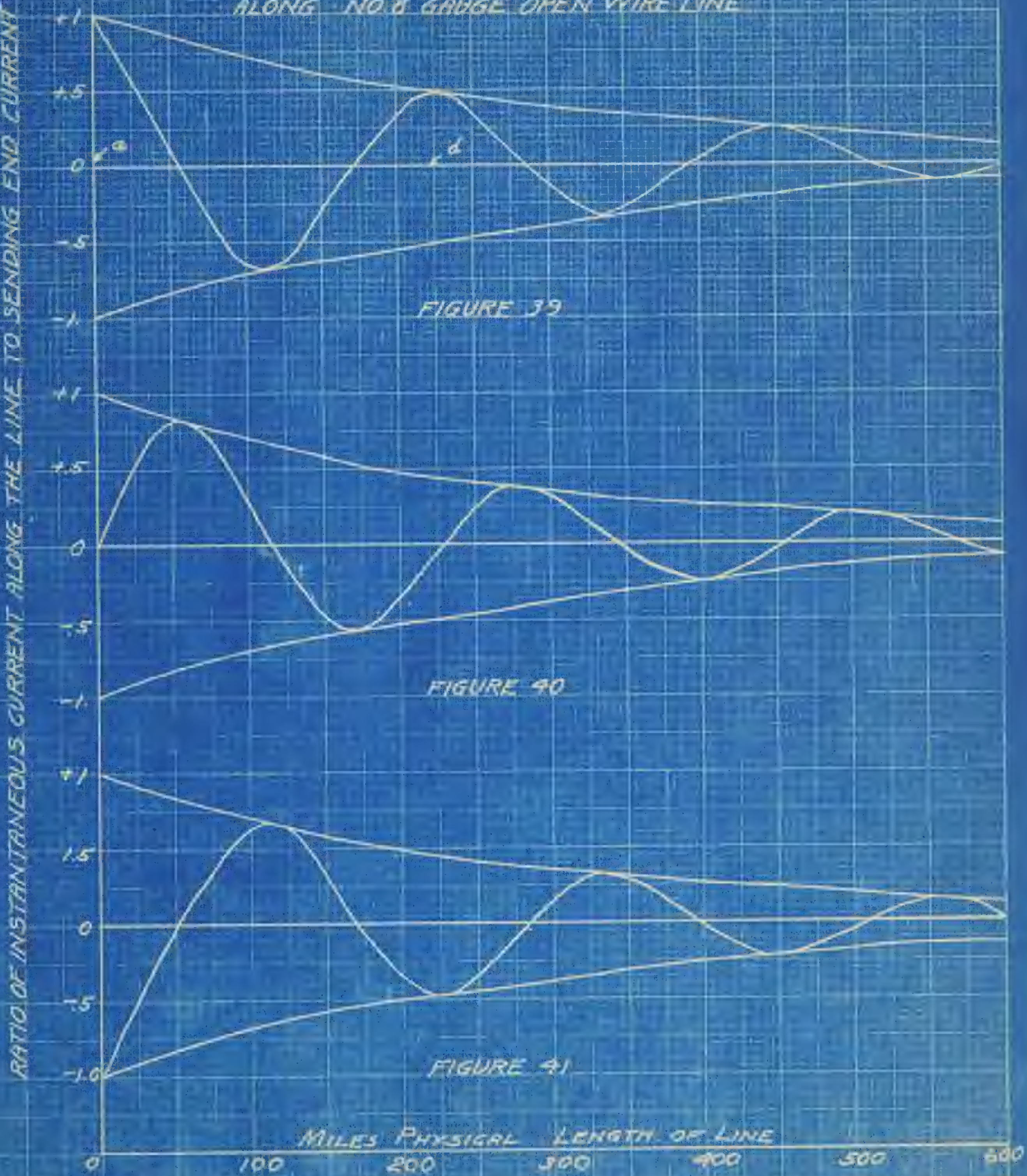


FIGURE 39

FIGURE 40

FIGURE 41

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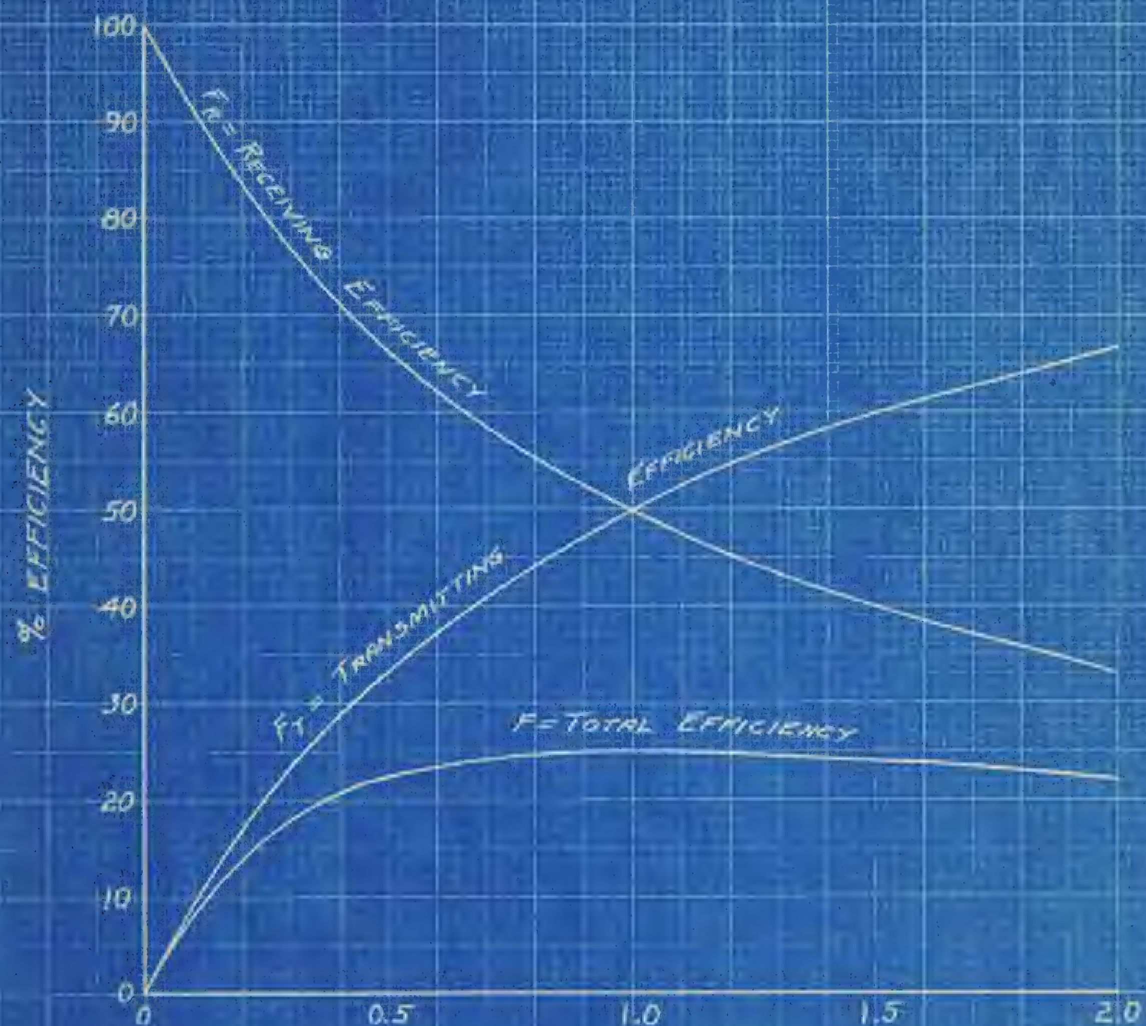
FIGURES TO ACCOMPANY NOTES ON FUNDAMENTALS OF TRANSMISSION.

RELATION BETWEEN TRANSMITTING, RECEIVING AND TOTAL EFFICIENCY IN IDEAL INVARIABLE SUBSTATION CIRCUIT.

FIG. 48




$\frac{F_T}{F_R}$ = RATIO $\frac{\text{TRANSMITTING EFFICIENCY}}{\text{RECEIVING EFFICIENCY}}$

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FIGURES ACCOMPANYING NOTES ON FUNDAMENTALS OF TRANSMISSION

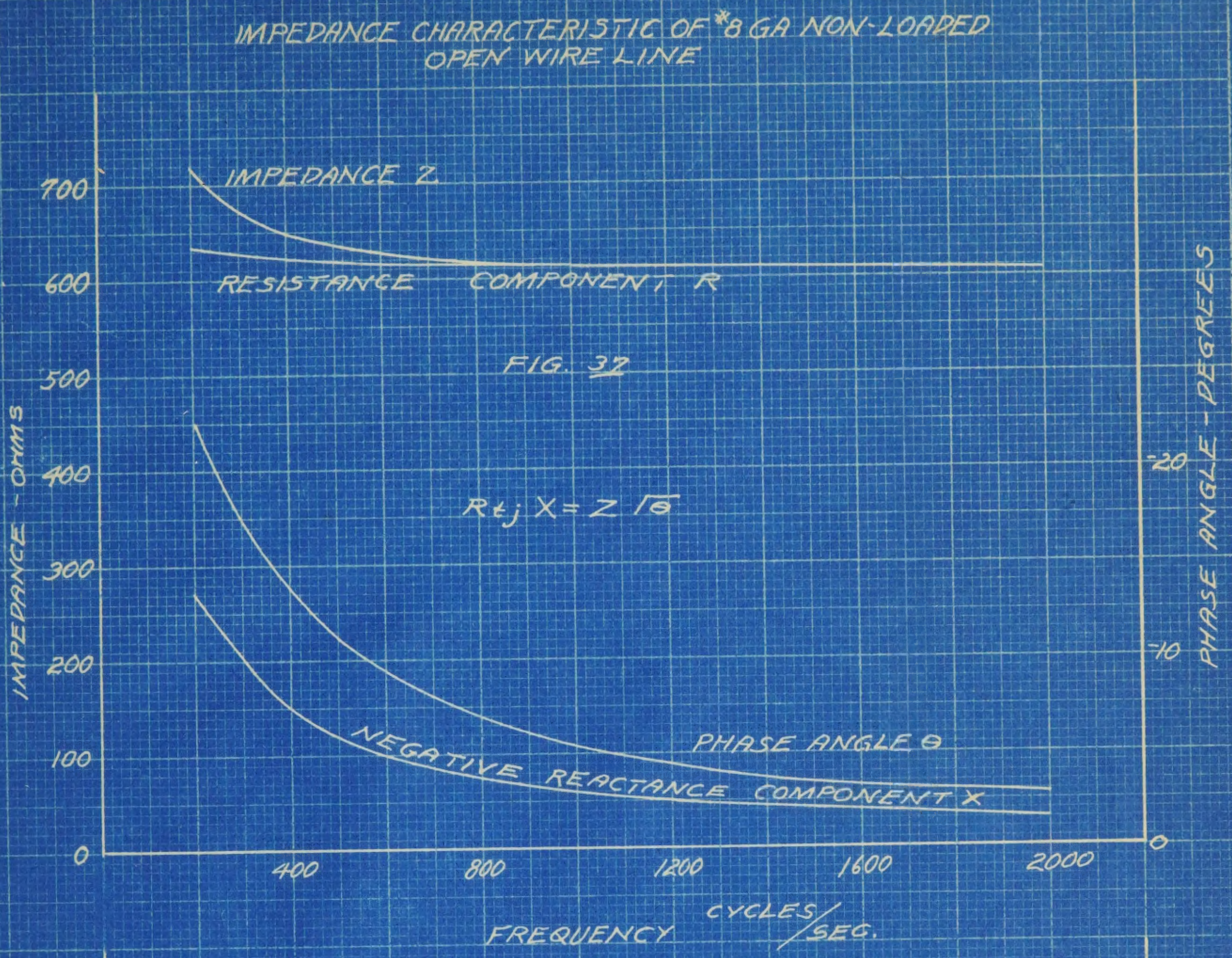


FIG. 37

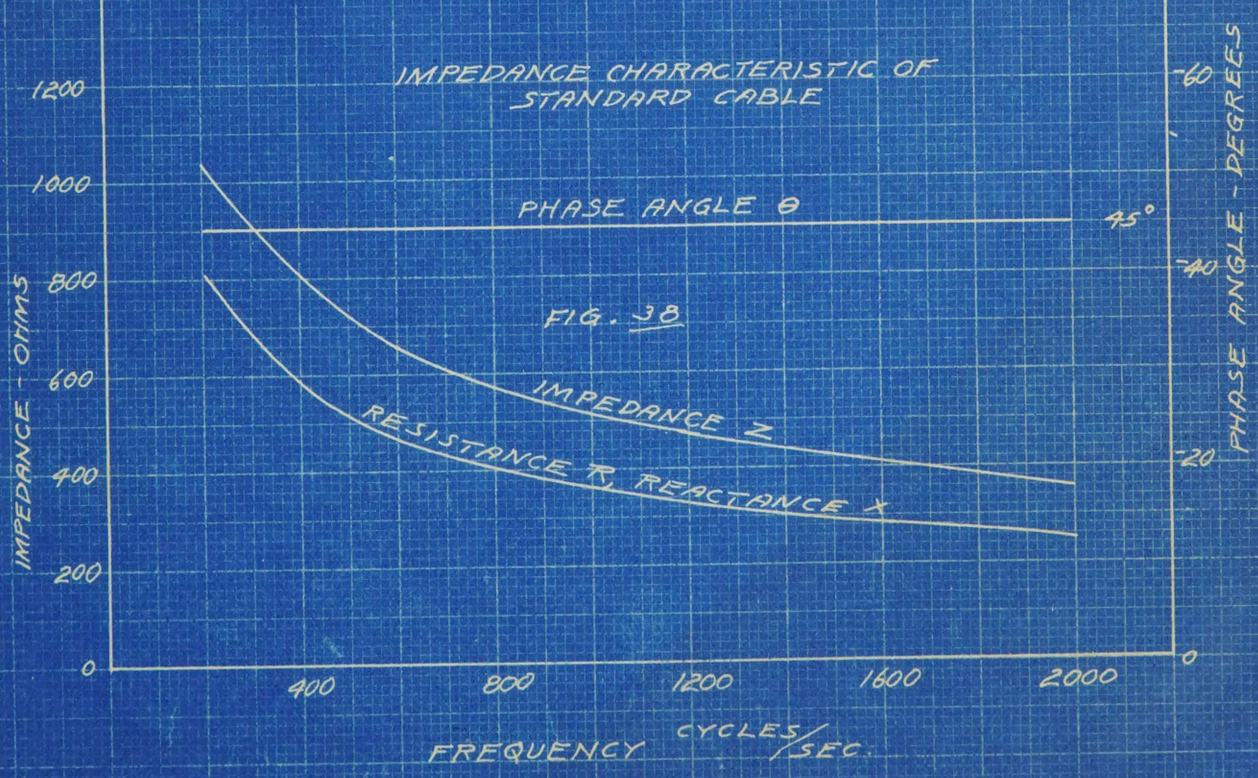


FIG. 38